1. (12%) Evaluate the improper integral $\int_{0^+}^{\infty} \frac{e^{-3x}}{\sqrt{2x}} dx$.

2. (12%) If electricity power failures occur according to a Poisson distribution with an average of 7 failures every 15 days, calculate the probability that there will be more than one failure during a particular day.

3. (13%) Let X, Y be independent random variables with probability density functions given by

$$f_X(t) = te^{-t}, \quad f_Y(t) = \frac{1}{2}t^2 e^{-t} \text{ for } t \geq 0,$$

and $f_X(t) = f_Y(t) = 0$ for $t < 0$. Find the probability density function $f_Z(t)$, where $Z = X + Y$.

4. (15%) Let X and Y be two independent random variables. X and Y both take value at $\{-1, 1\}$. Suppose we know their joint probability as follows:

$$P(X = 1, Y = -1) = \frac{8}{15}$$

$$P(X = -1, Y = -1) = \frac{4}{15}$$

$$P(X = 1, Y = 1) = \frac{2}{15}$$

$$P(X = -1, Y = 1) = \frac{1}{15}$$

Let $Z = \frac{X \cdot Y + 1}{2}$.

   (a) (3%) Find the range of Z and the prabability function of Z.

   (b) (3% each, 6% total) Find $\mathrm{E}(Z)$ and $\mathrm{Var}(Z)$。

   (c) (2% each, 6% total) Suppose $W_n \sim Z$ and are independent with each other. Let

   $$W = W_1 + W_2 + \cdots + W_{10}.$$

   Find $\mathrm{P}(W = 4)$, expectation $\mathrm{E}(W)$ and varance $\mathrm{Var}(W)$。

5. (12%) Let the random variable W denote the waiting time in a fast food restaurant. It is known that the average of W is 2 minutes and that W has the exponential distribution.

   (a) (3%) Write down the probability density function of W.

   (b) (3%) Find the probability $\mathrm{P}(1 < W < 3)$.

   (c) (3% each, 6% total) Find the expectation $\mathrm{E}(W)$ and the variance $\mathrm{Var}(W)$. Please write down your detail calculation.

6. (12%) Solve the differential equation $(t^2 + 2)y'(t) + (4t)y(t) = 1$ satisfying $y(0) = 2$.

7. (12%) Solve the differential equation $y'(t) = \sin t + \sin t(y(t))^2$. Find the general solution.

8. (12%) Let X be a continuous random variable with the probability density function $f_X(t) = \frac{1}{\sqrt{\pi}} e^{-t^2}$. We define a new random variable $W = X^3$. Find the corresponding probability density of the random variable W.