

1. (10%) Solve the differential equation
$$\begin{cases} y' + (\tan t)y = t \sin 2t, & -\frac{\pi}{2} < t < \frac{\pi}{2}, \\ y(0) = 1. \end{cases}$$

2. (15%) Solve the differential equation $\frac{dy}{dx} = e^y \sin x$.

3. (10%) A chemical reaction involves the collision of one molecule of a substance P with one molecule of a substance Q to produce one molecule of a new substance X. This is $P+Q \rightarrow X$. If $p, q, (p \neq q)$ are the initial concentrations of P and Q, respectively, and let $x(t)$ be the concentration of X at time t . Then $p - x(t)$ and $q - x(t)$ are the concentrations of P and Q at time t , respectively. The reaction is given by the equation

$$\frac{dx}{dt} = \alpha (p - x)(q - x),$$

where α is a positive constant. If $x(0) = 0$, solve $x(t)$.

4. (10%) Let X be a random variable, taking values in $\{1, 2\}$. Let $E(X) = \frac{5}{3}$.

(a) (5%) Find $P(X = 1)$ and $P(X = 2)$.

(b) (5%) Evaluate $\text{Var}(X)$.

5. (10%) Evaluate $\int_{-\infty}^{\infty} e^{-x^2+2bx+c} dx, b, c \in \mathbb{R}$. (You can use $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.)

6. (10%) Let X, Y be random variables such that $f_X(t) = e^{-t}, t \geq 0$ and $Y = 2X + 1$. Find $f_Y(t)$.

7. (10%) Let X, Y be two independent random variables both with the probability density $f_X(t) = f_Y(t) = \frac{1}{\sqrt{\pi}} e^{-t^2}, t \in \mathbb{R}$. Find the probability density function $f_W(t)$ of the random variable $W = (X + Y)^2$.

8. (10%) The average number of phone calls is 20 per hour. Find the probability for the event of at least one phone call within an interval of 3 minutes. (Assume the validity of Poisson process for this case.)

9. (15%) Let X be the random variable for the duration of telephone calls within a certain city with the probability density function

$$f_X(t) = \frac{2}{5} e^{-\frac{2t}{5}}, t > 0$$

where t denotes the duration in minutes of a randomly selected call.

(a) (4%) What percentage of calls last one minute or less ?

(b) (4%) What percentage of calls last between one and two minutes ?

(c) (4%) What percentage of calls last 3 minutes or more ?

(d) (3%) What is the average length of a call ?