991微甲08-12班期末考解答和評分標準

1. (7%) Evaluate $\displaystyle\int_0^{\ln 7} \frac{e^{2x}}{(e^x+1)^{\frac{1}{3}}}\,dx$.

Sol:

Let $u = e^x + 1$, $du = e^x dx$, $x = 0, u = 2, x = \ln 7, u = 8$.

$$\int_0^{ln7} \frac{e^{2x}}{(e^x+1)^{\frac{1}{3}}}\,dx = \int_2^8 \frac{u-1}{u^{\frac{1}{3}}}\,du \qquad \text{(3pt up)}$$

$$= \int_2^8 u^{\frac{2}{3}} - u^{\frac{1}{3}}\,du = \frac{3}{5}u^{\frac{5}{3}} - \frac{2}{3}u^{\frac{2}{3}}$$

$$= \frac{66}{5} + \frac{3}{10}2^{\frac{2}{3}}$$

2. (10%) Evaluate $\displaystyle\int_{0^+}^{\infty} \frac{4dx}{\sqrt{x}(4+x)}$. *Caution!* $\displaystyle\lim_{x\to 0^+} \frac{4}{\sqrt{x}(4+x)} = \infty$.

Sol:

Let $u = \sqrt{x}$, $du = \dfrac{1}{2\sqrt{x}}$, $x = 0^+, u = \sqrt{0^+}$

$$\lim_{b\to\infty} \int_{0^+}^{b} \frac{4dx}{\sqrt{x}(4+x)} = \lim_{b\to\infty} \int_{\sqrt{0^+}}^{\sqrt{b}} \frac{8}{4+u^2}\,du \qquad \text{(5pt up)}$$

Let $u = 2\tan\theta$, $du = 2\sec^2\theta d\theta$,

$$\lim_{b\to\infty} \int_{\sqrt{0^+}}^{\sqrt{b}} \frac{8}{4+u^2}\,du = \int_0^{\frac{\pi}{2}} 4\,d\theta = 2\pi$$

θ 變換好 7pt up

一點點計算錯誤扣一分, 上下限扣兩分。

3. (8%) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{3-5\sin x}$. *Hint*: Set $t = \tan\left(\frac{x}{2}\right)$ and express $\sin x$ and dx in terms of t.

Sol:

Wrong answer (you will get points even if the answer here is incorrect)Let $t = \tan(\dfrac{x}{2}) \Rightarrow$

$\sin x = \dfrac{2t}{1+t^2}$ *and* $dx = \dfrac{2}{1+t^2}\,dt$,

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{dx}{3-5\sin x} &= \int_0^1 \frac{1}{3-\frac{5\cdot 2t}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt \\
&= 2\int_0^1 \frac{1}{3t^2-10t+3}\,dt \quad (3\%) \\
&= 2\int_0^1 \frac{1}{(3t-1)(t-3)}\,dt \\
&= 2\int_0^1 \left(\frac{-\frac{3}{8}}{3t-1} + \frac{\frac{1}{8}}{t-3}\right) dt \\
&= \frac{1}{4}[-\ln|3t-1| + \ln|t-3|]\Big|_0^1 \\
&= -\frac{1}{4}\ln 3 \quad (5\%)
\end{aligned}$$

Correct answer

$$2\int_0^1 \frac{1}{3t^2-10t+3}\,dt = \lim_{a\to\frac{1}{3}^-} 2\int_0^a \frac{1}{3t^2-10t+3}\,dt + \lim_{b\to\frac{1}{3}^+} 2\int_b^1 \frac{1}{3t^2-10t+3}\,dt$$

Since

$$\begin{aligned}
\lim_{a\to\frac{1}{3}^-} 2\int_0^a \frac{1}{3t^2-10t+3}\,dt &= \lim_{a\to\frac{1}{3}^-} 2\int_0^a \left(\frac{-\frac{3}{8}}{3t-1} + \frac{\frac{1}{8}}{t-3}\right) dt \\
&= \lim_{a\to\frac{1}{3}^-} \frac{1}{4}[-\ln|3t-1| + \ln|t-3|]\Big|_0^a = \infty
\end{aligned}$$

and

$$\begin{aligned}
\lim_{b\to\frac{1}{3}^+} 2\int_b^1 \frac{1}{3t^2-10t+3}\,dt &= \lim_{b\to\frac{1}{3}^+} 2\int_b^1 \left(\frac{-\frac{3}{8}}{3t-1} + \frac{\frac{1}{8}}{t-3}\right) dt \\
&= \lim_{b\to\frac{1}{3}^+} \frac{1}{4}[-\ln|3t-1| + \ln|t-3|]\Big|_b^1 = -\infty
\end{aligned}$$

$\therefore 2\displaystyle\int_0^1 \frac{1}{3t^2-10t+3}\,dt$ does not exist.

4. (10%) Let Ω be the region enclosed by $y = \dfrac{2e^{\sqrt{x}}}{2x-1}$, $y=0$, $x=1$ and $x=4$. Find the volume generated by revolving Ω about the line $x = \dfrac{1}{2}$.

Sol:

$$\begin{aligned}
\int_1^4 2\pi(x-\frac{1}{2})\Big(\frac{2e^{\sqrt{x}}}{2x-1}\Big)\,dx(3\%) &= 2\pi\int_1^4 e^{\sqrt{x}}\,dx \\
&= 4\pi\int_1^2 ue^u\,du \qquad (u=\sqrt{x} \Rightarrow dx = 2u\,du) \\
&= 4\pi(ue^u - e^u)\Big|_1^2 = 4\pi e^2 \quad (7\%).
\end{aligned}$$

5. (10%) Evaluate $\int_1^2 \frac{x^2+4}{x^4+3x^3+2x^2}\,dx$.

Sol:

$$\frac{x^2+4}{x^4+3x^3+2x^2} = \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x+1} + \frac{d}{x+2} \text{ (4 pts)}$$

$\Rightarrow x^2+4 = ax(x+1)(x+2) + b(x+1)(x+2) + cx^2(x+2) + dx^2(x+1)$

Put $x=0$: $4 = 2b \;\Rightarrow\; b = 2$.

Put $x=-1$: $5 = c$.

Put $x=-2$: $8 = -4d \;\Rightarrow\; d = -2$.

Put $x=1$: $5 = 6a+6b+3c+2d \;\Rightarrow\; a = -3$. (3 pts)

$$\begin{aligned}\int_1^2 \frac{x^2+4}{x^4+3x^3+2x^2}\,dx &= \int_1^2 \Big(\frac{-3}{x} + \frac{2}{x^2} + \frac{5}{x+1} + \frac{-2}{x+2}\Big)\,dx \\ &= -3\ln x\Big|_1^2 + (-2)\frac{1}{x}\Big|_1^2 + 5\ln(x+1)\Big|_1^2 - 2\ln(x+2)\Big|_1^2 \\ &= -3\ln 2 + 1 + 5(\ln 3 - \ln 2) - 2(\ln 4 - \ln 3) \\ &= 1 + 7\ln 3 - 12\ln 2. \text{ (3 pts)}\end{aligned}$$

6. (15% total)

   (a) Find $\int \sec^3 x\,dx$, given $\int \sec x = \ln|\sec x + \tan x| + \text{a constant}$. (7%)

   (b) The curve $y = \ln x$, $0 < x < 1$, is rotated about y-axis. Find the area of the resulting surface. (8%)

Sol:

   (a) step1: Using integration by parts yields

$$\int \sec^3 x\,dx = \tan x \sec x - \int \tan^2 x \sec x\,dx = \tan x \sec x - \int (\sec^2 x - 1)\sec x\,dx.$$

   step2: By rearrangement of the above result, it follows that

$$\int \sec^3 x\,dx = \frac{1}{2}\Big(\sec x \tan x + \ln\big|\sec x + \tan x\big|\Big) + C,$$

   where C is an arbitrary constant.

   (b) step3: The area of the resulting surface is

$$\int \text{circumference} * ds = \int_0^1 (2\pi x)\sqrt{1+(y')^2}\,dx = 2\pi\int_0^1 \sqrt{1+x^2}\,dx.$$

   step4: Let $x = \tan\theta$, then $\frac{dx}{d\theta} = \sec^2\theta$ and the previous integral thus becomes

$$2\pi\int_0^{\frac{\pi}{4}} (\sec\theta)\sec^2\theta\,d\theta.$$

step5: From (a), one can evaluate

$$2\pi \int_0^{\frac{\pi}{4}} \sec^3\theta\, d\theta = \sqrt{2}\pi + \pi \ln(\sqrt{2}+1),$$

which completes (b).

Grading Policy:

1. 4 points for step1.

2. 3 points for step2 with total 7 points for (a).

3. 4 points for step3.

4. 2 points for step4.

5. 2 points for step5 with total 8 points for (b).

7. (10%) Solve $(x^2+1)^2\dfrac{dy}{dx} + 3x(x^2+1)y = 2x$ with the initial condition $y(0) = 3$.

Sol:

step1: First we derive the integrating factor and take $C = 0$ to make it simple,

$$I(x) = e^{\int \frac{3x}{x^2+1}dx} = (x^2+1)^{\frac{3}{2}}. \quad \text{(5 pts)}$$

step2: By multiplying the above integrating factor, the original ODE can be rearranged as

$((x^2+1)^{\frac{3}{2}}y)' = 2x(x^2+1)^{-\frac{1}{2}}$

$\Rightarrow ((x^2+1)^{\frac{3}{2}}y) = 2(x^2+1)^{\frac{1}{2}} + C'$

$\Rightarrow ((x^2+1)^{\frac{3}{2}}y) = 2(x^2+1)^{\frac{1}{2}} + 1$ (5 pts)

8. (10%) Solve $\dfrac{dy}{dx} = \dfrac{y^2\sin x}{1+y^3}$, $y(0) = 1$.

Sol:

$$\frac{1+y^3}{y^2}dy = \sin x dx \qquad \text{(5 pts)}$$

$$\int \frac{1+y^3}{y^2}dy = \int \sin x dx$$

$$\frac{-1}{y} + \frac{y^2}{2} = -\cos x + c \qquad \text{(3 pts)}$$

Since $y(0) = 1$, we have

$$-1 + \frac{1}{2} = -1 + c$$

$$\frac{1}{2} = c \qquad \text{(2 pts)}$$

Hence, the solution is

$$\frac{y^2}{2} - \frac{1}{y} + \cos x = \frac{1}{2}.$$

9. (10%) Consider the parametric curve with $x'(t) = \sqrt{3}t$, $y'(t) = 2t\sqrt{t-t^2}$, $0 \le t \le 1$. Find the arc length of this curve.

Sol:

$$\begin{aligned} \text{Length } &= \int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt \quad \text{(1 pt)} \\ &= \int_0^1 t\sqrt{-4t^2 + 4t + 3}\, dt \\ &= \int_0^1 2t\sqrt{1 - (t - \frac{1}{2})^2}\, dt \quad \text{(2 pts)} \end{aligned}$$

$$\text{Let } \theta = \arcsin(t - \frac{1}{2}) \Rightarrow t = \frac{1}{2} + \sin\theta, \text{ and } dt = \cos\theta d\theta \quad \text{(1 pt)}$$

$$\begin{aligned} \text{Length } &= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} (2\sin\theta + 1)\cos\theta\cos\theta\, d\theta \quad \text{(2 pts.)} \\ &= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} 2\sin\theta\cos^2\theta d\theta + \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos^2\theta\, d\theta \\ &= (-\frac{2}{3}\cos^3\theta)\Big|_{-\frac{\pi}{6}}^{\frac{\pi}{6}} + (\frac{\theta}{2} + \frac{\sin 2\theta}{4})\Big|_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \quad \text{(1 pt for each term)} \\ &= 0 + \frac{\pi}{6} + \frac{\sqrt{3}}{4} \\ &= \frac{\pi}{6} + \frac{\sqrt{3}}{4} \quad \text{(2 pts)} \end{aligned}$$

10. (10%) Find the surface area of the solid generated by revolving the cardioid $r = 1 + \sin\theta$ about y-axis.

Sol:

$$
\begin{aligned}
A &= \int_0^{\pi} 2\pi x\, ds \\
&= \int_{-\pi/2}^{\pi/2} 2\pi r\cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \qquad (x = r\cos\theta.) \\
&= 2\pi \int_{-\pi/2}^{\pi/2} (1+\sin\theta)\cos\theta\sqrt{2+2\sin\theta}\, d\theta \\
&= 2\pi \int_0^2 t\sqrt{2t}\, dt \qquad (\text{Set } t = 1+\sin\theta.) \\
&= \frac{32}{5}\pi.
\end{aligned}
$$

Grading Policy:

1. Write down or derive $ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$. (3 pt)
2. Insert $r(\theta)$ (1 pt) and the corresponding range of θ (2 pts).
3. Evaluate this integral by using any proper techniques and taking correct steps. (2 pts. You will get all of these 2 points if you only make a mistake in inserting a range of θ in previous steps.)
4. Give the correct value of the surface area. (2 pts)