1. (10%) Evaluate the intergal

$\int x^2 \ln(x^2-1)\,dx =$ ________________________.

Sol:

$$\begin{aligned}\int x^2 \ln(x^2-1)dx &= \frac{x^3}{3}\ln(x^2-1) - \int \frac{x^3}{3}\frac{2x}{x^2-1}dx \quad \text{(4 pts)}\\ &= \frac{x^3}{3}\ln(x^2-1) - \frac{2}{3}\int \left[(x^2+1) + \frac{1}{x^2-1}\right]dx \quad \text{(6 pts)}\\ &= \frac{x^3}{3}\ln(x^2-1) - \frac{2}{9}x^3 - \frac{2}{3}x + \frac{1}{3}\ln|x+1| - \frac{1}{3}\ln|x-1| + C\end{aligned}$$

2. (10%) Solve the initial value problem $y' = xe^{-\sin x} - y\cos x,\ y(0) = \frac{1}{2}$.

Ans: ________________________.

Sol:

$$\begin{aligned} I(x) &= e^{\sin x} \quad \text{(4 pts)}\\ \Rightarrow ye^{\sin x} &= \frac{x^2}{2} + C \quad \text{(6 pts)}\\ \Rightarrow y &= e^{-\sin x}\left(\frac{x^2}{2} + \frac{1}{2}\right)\end{aligned}$$

3. (10%) Let $y = y(x)$ be a differentiable function on $[0,\infty)$ which satisfies $y(x) = \frac{1}{2} - \int_0^{\sqrt{x}} t^3 y(t^2)\,dt$.

   (a) Determine a first order differential equation with an initial condition that $y(x)$ satisfies.

   Ans: ________________ with initial condition ________.

   (b) Solve this differential equation.

   Ans: $y(x) =$ ________________________.

Sol:

(a) step1: By chain rule and fundamental theorem of calculus I:

$$y' = -\frac{x^{\frac{3}{2}y(x)}}{2\sqrt{x}} = -\frac{1}{2}xy(x). \quad \text{(3 pts)}$$

step2: By pluging x=0 into the integral equation, we have

$$y(0) = \frac{1}{2}. \quad \text{(2 pts)}$$

(b) step3: Note that the differential equation in (a) is separable, it follows that

$$\int \frac{1}{y} dy = \int -\frac{1}{2} x dx$$

which becomes

$$\ln y = -\frac{x^2}{4} + C.$$

And the initial condition yields the result $C = \ln \frac{1}{2}$. (5 pts)

4. (10%) Evaluate the improper integral $\displaystyle\int_1^{\infty} \frac{x}{x^8+4}\, dx =$ ____________________.

Sol:

$$\begin{aligned}
\int_1^{\infty} \frac{x}{x^8+4} dx &= \lim_{t\to\infty} \int_1^t \frac{x}{x^8+4} dx && \text{(1 pt)} \\
(\text{let } u = x^2) \quad &= \lim_{t\to\infty} \frac{1}{2} \int_1^{t^2} \frac{du}{u^4+4} \\
&= \lim_{t\to\infty} \frac{1}{2} \int_1^{t^2} \frac{du}{(u^2+2u+2)(u^2-2u+2)} && \text{(2 pts)} \\
&= \lim_{t\to\infty} \frac{1}{2} \int_1^{t^2} \frac{1}{8} \left(\frac{u+2}{u^2+2u+2} - \frac{u-2}{u^2-2u+2} \right) du && \text{(2 pts)} \\
&= \lim_{t\to\infty} \frac{1}{16} \int_1^{t^2} \left(\frac{1}{2} \frac{2u+2}{u^2+2u+2} + \frac{1}{(u+1)^2+1} - \frac{1}{2} \frac{2u-2}{u^2-2u+2} + \frac{1}{(u-1)^2+1} \right) du \\
\begin{pmatrix} y = u^2+2u+2 \\ z = u^2-2u+2 \end{pmatrix} \quad &= \lim_{t\to\infty} \Bigg(\frac{1}{16} \int_5^{t^4+2t^2+2} \frac{1}{2} \frac{dy}{y} - \frac{1}{16} \int_1^{t^4-2t^2+2} \frac{1}{2} \frac{dz}{z} \\
&\quad + \frac{1}{16} \tan^{-1}(u+1) \Big|_1^{t^2} + \frac{1}{16} \tan^{-1}(u+1) \Big|_1^{t^2} \Bigg) \\
&= \lim_{t\to\infty} \Bigg(\frac{1}{32} \ln y \Big|_5^{t^4+2t^2+2} - \frac{1}{32} \ln z \Big|_1^{t^4-2t^2+2} \\
&\quad + \frac{1}{16} (\tan^{-1}(t^2+1) + \tan^{-1}(t^2-1) - \tan^{-1} 2 - \tan^{-1} 0) \Bigg) \\
&= \lim_{t\to\infty} \left(\frac{1}{32} \left(\ln \frac{t^4+2t^2+2}{t^4-2t^2+2} - \ln 5 \right) + \frac{1}{16} (\tan^{-1}(t^2+1) + \tan^{-1}(t^2-1) - \tan^{-1} 2) \right) \\
&= \frac{1}{32} (\ln 1 - \ln 5) + \frac{1}{16} \left(\frac{\pi}{2} + \frac{\pi}{2} - \tan^{-1} 2 \right) \\
&= \frac{1}{16} \pi - \frac{1}{32} \ln 5 - \frac{1}{16} \tan^{-1} 2 && \text{(5 pts)}
\end{aligned}$$

5. (15%) Let R be the region $\left\{ (x, y) | 0 \le y \le \frac{1}{x^p},\ x \ge 1 \right\}$.

(a) Rotate R about the line $y = -a,\ a > 0$. For what values of p is the volume of the resulting solid finite? Ans: __________. For such p, the volume is ________________.

(b) Rotate R about the line $y = 0$. For what values of p is the resulting surface area finite?

Ans: ________________________________.

Sol: (15 % = 6+6+3)

(a) The volume is

(1)

$$V = \int_1^{\infty} \left[\pi(\frac{1}{x^p} + a)^2 - \pi a^2\right] dx = \pi \int_1^{\infty} \left(\frac{1}{x^{2p}} + \frac{2a}{x^p}\right) dx$$

It converges if and only if both integrals converge. Thus, the volume of the resulting solid is finite for

(2) p > 1, ====> $\boxed{\textbf{6 points}}$ and the volume is

$$V = \pi\left(\frac{1}{2p-1} + \frac{2a}{p-1}\right) \qquad \boxed{\text{3 points}}$$

(b) The surface area is

(3)

$$2\pi \int_1^{\infty} \frac{1}{x^p} \sqrt{1 + (\frac{-p}{x^{p+1}})^2}\, dx$$

By Comparison Test

$$1 \leq \sqrt{1 + (...)^2} \leq c$$

or Limit Comparison Test the above integral converges if and only if

$$\int_1^{\infty} x^{-p}\, dx \qquad \text{converges.}$$

Thus the resulting surface area is finite for

(4) p > 1, ====> $\boxed{\textbf{6 points}}$

Remark:

1. Please check the calculation that if they derive the expressions (1) and /or (3) first. If there are no (1) and / or (3), even the answers are correct, they get **zero points.**

2. Give 4 points if the answer of the first p in (a) is incorrect but equation (1) is correct.

3. Give 4 points if the answer of the second p in (b) is incorrect but equation (3) is correct.

6. (15%) (a) Sketch the curve with the polar equation $r = 1 + 2\cos 2\theta$.

(b) Find the intersections of $r = 1$ and $r = 1 + 2\cos 2\theta$.

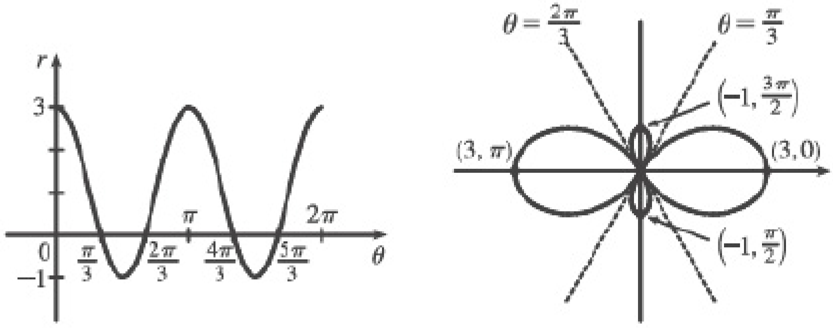
Ans: ____.

(c) Find the area of the region that is inside the curve $r = 1$ and outside the curve $r = 1 + 2\cos 2\theta$. Ans: ____.

Sol:

(a) (4pts)

$r = 1 + 2\cos 2\theta$



(b) Solve the equation:

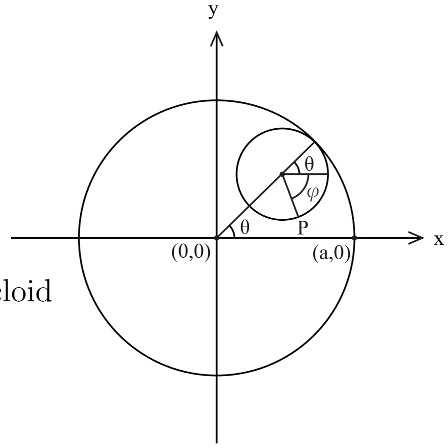
$$1 + 2\cos 2\theta = 1$$
$$\cos 2\theta = 0$$
$$\theta = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}, \pm\frac{\pi}{2}. \qquad \text{(4pts)}$$

(c) By symmetry, the area is:

$$= 4\Big[\frac{\pi}{8} - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2}(1 + 2\cos 2\theta)d\theta\Big] \qquad \text{(3pts)}$$
$$= 4\Big[\frac{\pi}{8} - \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 1 + 4\cos 2\theta + 4\cos^2 2\theta d\theta\Big]$$
$$= 4\Big[\frac{\pi}{8} - \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 1 + 4\cos 2\theta + 4\cdot\frac{1 + \cos 4\theta}{2} d\theta\Big] \qquad \text{(2pts)}$$
$$= 4\Big[\frac{\pi}{8} - (3\theta + 2\sin 2\theta + \frac{1}{2}\sin 4\theta)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}\Big]$$
$$= 4 - \pi. \qquad \text{(2pts)}$$

7. (15%) A hypocycloid is a curve traced out by a fixed point P on a circle C of radius b as C rolls on the inside of a circle with center $(0, 0)$ and radius a $(a > b)$. Assume that the initial

position of P is $(a, 0)$ and the parameter θ is chosen as in the figure (i.e. θ is the angle between the line connecting the centers of the circles and the x-axis.)



(a) Let φ be the angle as shown in the figure on the right.

Express φ in terms of θ. Ans: ______________.

(b) It is known that the parametric equation of the hypocycloid

is $\begin{cases} x(\theta) = C_1 \cos\theta + C_2 \cos\left(\dfrac{a-b}{b}\theta\right) \\ y(\theta) = C_3 \sin\theta + C_4 \sin\left(\dfrac{a-b}{b}\theta\right), \end{cases}$

where $C_1,\ C_2,\ C_3,$ and C_4 are constants.

Find $C_1 =$ ________, $C_2 =$ ________, $C_3 =$ ________, $C_4 =$ ________.

(c) For $a = 7,\ b = 3$ and $0 \le \theta \le 2\pi$, the arc length of the hypocycloid is __________.

Sol:

(a) (3pt) $\varphi = \underline{-\dfrac{a-b}{b}\theta}$ or $\underline{\dfrac{a-b}{b}\theta}$ The two answers are both ok.

(b) (4pt) $C_1 = \underline{\ a-b\ }$, $C_2 = \underline{\ b\ }$, $C_3 = \underline{\ a-b\ }$, $C_4 = \underline{\ -b\ }$. Every correct coefficient deserves 1pt. Or you may write as the following formula in parameter θ.

$$x(\theta) = (a-b)\cos\theta + b\cos\frac{(a-b)\theta}{b}$$
$$y(\theta) = (a-b)\sin\theta - b\sin\frac{(a-b)\theta}{b}$$

(c) (8pt) Each step in the following deserves 2pt. If the students have some wrong coefficients in (b), there will still be some process earning part of the points.

For $a = 7$, $b = 3$ and $0 \le \theta \le 2\pi$, the arc length of the hypocycloid is?

$\boxed{\text{Step I}}$ Write down the formula and calculate the derivatives of x and y (2pt)

$$L = \int_0^{2\pi} \sqrt{x'(\theta)^2 + y'(\theta)^2}\, d\theta$$

$\boxed{\text{Step II}}$ Simplify the integrant (2pt)

$$= \int_0^{2\pi} \sqrt{32 - 32\cos\frac{7}{3}\theta}\, d\theta$$

$\boxed{\text{Step III}}$ Use the half-angle formula (2pt)

$$= \int_0^{2\pi} 8\sqrt{\sin^2\frac{7}{6}\theta}\, d\theta = \int_0^{2\pi} 8\left|\sin\frac{7}{6}\theta\right|\, d\theta$$

$\boxed{\text{Step IV}}$ Divide $[0, 2\pi]$ into subintervals to calculate the integral of absolute value of sine (2pt)

$$= 8\left[\int_0^{6\pi/7} \sin\frac{7}{6}\theta\, d\theta - \int_{6\pi/7}^{12\pi/7} \sin\frac{7}{6}\theta\, d\theta + \int_{12\pi/7}^{14\pi/7} \sin\frac{7}{6}\theta\, d\theta\right] = 8\cdot\frac{6}{7}(2+2+1-1/2) = \frac{216}{7}$$

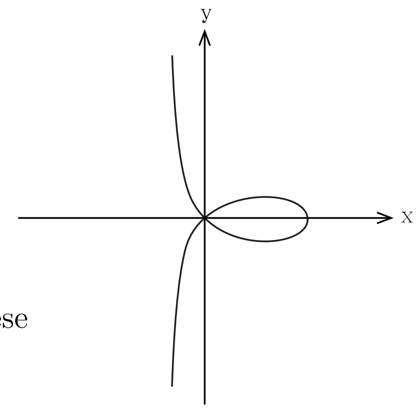
Some common mistake and the points:

- something wrong happens in calculation of x' and y': 0pt.
- use wrong coefficients in (b) and work out the outcome of Step II: 3pts.
- since the integrant is not in period $\pi/2$ or π, so this is wrong:

$$(X)\quad \int_0^{2\pi} \sqrt{x'^2 + y'^2}\, d\theta = 2\int_0^{\pi} \sqrt{x'^2 + y'^2}\, d\theta$$
$$(X)\quad \int_0^{2\pi} \sqrt{x'^2 + y'^2}\, d\theta = 4\int_0^{\pi/2} \sqrt{x'^2 + y'^2}\, d\theta$$

  if the process is correct, it deserves 3pts.

8. (15%) Let C be the curve defined by $x(x^2 + y^2) = (x^2 - y^2)$.



(a) Find parametric equations of C. (Hint: Let $y = tx$.)

Ans: $x =$ ________________,

$y =$ ________________.

(b) There are two tangents at the origin. The equations of these two tangents are ________________.

(c) The area of the region enclosed by the loop of this curve is ________________.

Sol:

(a)

$$x^3(1 + t^2) = x^2(1 - t^2)$$

$$x = 0, (x, y) = (0, 0)$$

$$x \neq 0, x = \frac{1 - t^2}{1 + t^2}, y = t\frac{1 - t^2}{1 + t^2}$$

$$\Rightarrow x = \frac{1 - t^2}{1 + t^2}, y = t\frac{1 - t^2}{1 + t^2}, \quad \text{(2 pts each)}$$

(b)

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{-2t^4-4t^2+2}{(1+t^2)^2}}{\frac{-4t}{(1+t^2)^2}} = \frac{-2t^4-4t^2+2}{-4t}, \quad \text{(3 pts)}$$

$$(x,y) = (0,0) \Leftrightarrow t = 1 \text{or} t = -1$$

$$t = 1, \frac{dy}{dx} = 1, \text{tanget line} : y = x, \quad \text{(1pt)}$$
$$t = -1, \frac{dy}{dx} = -1, \text{tangent line} : y = -x, \quad \text{(1pt)}$$

(c) if we consider the graph above the x-axis, say $y = f(x) > 0$ for $0 \le x \le 1$

$$\because x = \frac{1-t^2}{1+t^2} \Rightarrow t^2(1+x) = (1-x) \Rightarrow t = \sqrt{\frac{1-x}{1+x}}$$

$$f(x) = y = tx = x\sqrt{\frac{1-x}{1+x}}$$

$$A = 2\int_0^1 y\,dx = 2\int_0^1 f(x)\,dx = 2\int_0^1 x\sqrt{\frac{1-x}{1+x}}\,dx = 2\int_0^1 \frac{x\sqrt{1-x^2}}{1+x}\,dx$$

$$(\text{let } x = \sin\theta, dx = \cos\theta d\theta, \sqrt{1-x^2} = \cos\theta)$$

$$A = 2\int_0^{\frac{\pi}{2}} \frac{\sin\theta\cos\theta}{1+\sin\theta}\cos\theta\,d\theta = 2\int_0^{\frac{\pi}{2}} \frac{\sin\theta(1-\sin^2\theta)}{1+\sin\theta}\,d\theta = 2\int_0^{\frac{\pi}{2}} (1-\sin\theta)\sin\theta\,d\theta$$
$$= 2\int_0^{\frac{\pi}{2}} (\sin\theta - \sin^2\theta)\,d\theta = 2\int_0^{\frac{\pi}{2}} (\sin\theta - \frac{1}{2} + \frac{1}{2}\cos 2\theta)\,d\theta$$
$$= 2(-\cos\theta - \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta)\Big|_0^{\frac{\pi}{2}} = 2(-\frac{\pi}{4} - (-1)) = 2 - \frac{\pi}{2}$$

You should get 1 points if you have written down the range of integration correctly.

2 points more if the integrand is valid.

Finally, 3 points for the answer.