982微積分甲08-13班期中考解答與評分標準

1. (15%) Find the interval of x such that the power series $\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{\ln(k+1)}$ converges.

Sol:

(i)

$$\begin{aligned}\lim_{k\to\infty} \frac{|1/\ln(k+2)|}{|1/\ln(k+1)|} &= \lim_{k\to\infty} \frac{\ln(k+1)}{\ln(k+2)} \\ &= \lim_{k\to\infty} \frac{1/(k+1)}{1/(k+2)} \quad \text{(l'Hospital's rule)} \\ &= 1.\end{aligned}$$

So the radius of convergence is $\frac{1}{1} = 1$.

(ii) For $x = 1$,

$$\sum_{k=1}^{\infty} \frac{x^k}{\ln(k+1)} = \sum_{k=1}^{\infty} \frac{1}{\ln(k+1)}.$$

Since $\frac{1}{\ln(k+1)} > \frac{1}{(k+1)}$ for each $k \in \mathbb{N}$ and $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(k+1)}$ diverges, $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\ln(k+1)}$ diverges by comparison test.

For $x = -1$

$$\sum_{k=1}^{\infty} \frac{x^k}{\ln(k+1)} = \sum_{k=1}^{\infty} \frac{(-1)^k}{\ln(k+1)}.$$

Since $\frac{1}{\ln(k+1)} \searrow 0$ as $k \to \infty$, $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{\ln(k+1)}$ converges by the alternating series test.

From (i) and (ii), the interval of convergence is $[-1, 1)$.

評分標準:

(1) radius of convergence $\to 9$

- Knowing to use ratio test or root test $\to 3$
- Getting some results from one of the tests above but not to the extent of recognizing 1 as the radius of convergence $\to 4 \sim 7$

(2) Discussion of $x = 1 \to 3$

(3) Discussion of $x = -1 \to 3$

2. (10%) (a) Find the Maclaurin series for $f(y) = \sin y$.

(b) Evaluate $\int_0^{\frac{\pi}{2}} \sin(\cos x)\, dx$ correct to within an error of 0.01.

Sol:

(a)

$$f(y) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} y^n$$

since $\frac{d^{4k}}{dy^{4k}} \sin y = \sin y$, $\frac{d^{4k+1}}{dy^{4k+1}} \sin y = \cos y$, $\frac{d^{4k+2}}{dy^{4k+2}} \sin y = -\sin y$, $\frac{d^{4k+3}}{dy^{4k+3}} \sin y = -\cos y$

we have

$f^{(4k)}(0) = 0,\ f^{(4k+1)}(0) = 1,\ f^{(4k+2)}(0) = 0,\ f^{(4k+3)}(0) = -1,\ k = 0, 1, 2, 3, \cdots$ (2pts)

$$f(y) = \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n+1}}{(2n+1)!} \quad \text{(2pts)}$$

(b) since the radius of convergence is ∞, and $-1 \le \cos x \le 1$

$$\Rightarrow \sin(\cos x) = \cos x - \frac{\cos^3 x}{3!} + \frac{\cos^5 x}{5!} - \cdots \quad \text{(1pts)}$$

$$\int_0^{\frac{\pi}{2}} \sin(\cos x) dx = \sum_{n=0}^{\infty} \int_0^{\frac{\pi}{2}} \frac{(-1)^n \cos^{2n+1} x}{(2n+1)!} dx = \sum_{n=0}^{\infty} (-1)^n a_n$$

$1 > \cos x > 0$ for $0 < x < \frac{\pi}{2}$ thus a_n is positive and

$$\frac{\cos^{2n-1} x}{(2n-1)!} > \frac{\cos^{2n+1} x}{(2n+1)!} \Rightarrow \int_0^{\frac{\pi}{2}} \frac{\cos^{2n-1} x}{(2n-1)!} dx > \int_0^{\frac{\pi}{2}} \frac{\cos^{2n+1} x}{(2n+1)!} dx$$

$$0 \le a_n \le \frac{\pi}{2(2n+1)!}$$

$\Rightarrow a_n$ decreasing to 0, we can apply alternating series test,

$$a_3 = \int_0^{\frac{\pi}{2}} \frac{\cos^5 x}{120} dx = \int_0^{\frac{\pi}{2}} \frac{\cos x(1 - \sin^2 x)^2}{120} dx = \frac{1}{120}(\sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x)\Big|_0^{\frac{\pi}{2}}$$

$$= \frac{1}{225} < 0.01$$

only have to compute $a_0 - a_1$,error is less then 0.01 (3pts)

$$a_0 = \int_0^{\frac{\pi}{2}} \cos x dx = \sin x\Big|_0^{\frac{\pi}{2}} = 1$$

$$a_1 = \int_0^{\frac{\pi}{2}} \frac{\cos^3 x}{6} dx = \int_0^{\frac{\pi}{2}} \frac{\cos x(1-\sin^2 x)}{6} dx = (\frac{\sin x}{6} - \frac{\sin^3 x}{18})\Big|_0^{\frac{\pi}{2}} = \frac{1}{9}$$

$$a_0 - a_1 = \frac{8}{9} \text{ (2pts)}$$

3. (10%) Let $z = y + f(x^2 - y^2)$ and f be a differentiable function in one variable. Find the value of $y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y}$ when $x = a$ and $y = b$.

Sol:

$\frac{\partial z}{\partial x} = 0 + 2xf'(x^2 - y^2)$, $\frac{\partial z}{\partial y} = 1 - 2yf'(x^2 - y^2)$

$y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 2xyf'(x^2 - y^2) + x - 2xyf'(x^2 - y^2) = x$

So $y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y}\Big|_{(a,b)} = x\Big|_{(a,b)} = a$.

評分標準:

z 對 x 作偏微分: 3分

z 對 y 作偏微分: 3分

將上述二式併入所求式子: 3分

將 (a, b) 代入上式得其答案: 1分

其餘錯誤酌量給分

4. (10%) Let $f(x) = \ln(5 - x)$.

   (a) Find the power series representation for $f(x)$ at $x = 0$.

   (b) Find $f^{(n)}(0)$.

Sol:

(a) $\ln(1-t) = -t - \frac{t^2}{2} - ... = -\sum_{n=1}^{\infty} \frac{t^n}{n}$ (3 points)

and $\ln(5-x) = \ln 5 + \ln(1 - \frac{x}{5}) = \ln 5 - \sum_{n=1}^{\infty} \frac{(\frac{x}{5})^n}{n} = \ln 5 - \sum_{n=1}^{\infty} \frac{x^n}{n5^n}$ (3 points)

(b) By (a) $\Rightarrow f(0) = \ln 5$

$$f^{(n)}(0) = -n! \cdot \frac{1}{n5^n} = -\frac{(n-1)!}{5^n}, \quad n \geq 1. \qquad \text{(4 points)}$$

另一種:

(a) $\ln(5-x) = \int \frac{1}{5-t} dt$

$$= \frac{1}{5} \int \frac{1}{1-\frac{t}{5}} dt$$

$$= \frac{1}{5} \int \sum_{k=0}^{\infty} (\frac{t}{5})^k dt \qquad \text{(2 points)}$$

$$= \Big[\sum_{k=0}^{\infty} (\frac{x}{5})^{k+1} \frac{1}{k+1}\Big] + C \qquad \text{(2 points)}$$

choose $x = 0$, then $\ln 5 = C$, so $\ln(5-x) = \ln 5 - \sum_{n=1}^{\infty} \frac{x^n}{n5^n}$ (2 points)

(b) $c_n = \frac{f^{(n)}(0)}{n!}$ (2 points)

$f^{(n)}(0) = -\frac{(n-1)!}{5^n}, \quad n \geq 1$ (2 points)

$f(0) = \ln 5$

5. (13%) Let $\mathbf{r}(t) = \langle t^2, \frac{2}{3}t^3, t \rangle$.

(a) Find the arc length of $\mathbf{r}(t)$ from $t = 0$ to $t = 5$.

(b) Find the curvature of $\mathbf{r}(t)$ at $t = 4$.

Sol:

$\mathbf{r}'(t) =< 2t, 2t^2, 1 >$, $\mathbf{r}''(t) =< 2, 4t, 0 >$

(in (b) first solution 1%)

(a)

$$|\mathbf{r}'(t)| = \sqrt{4t^2 + 4t^4 + 1} \ (3\%)$$

$$= 2t^2 + 1 \ (2\%)$$

$$L = \int_0^5 |\mathbf{r}'(t)|dt \ = \int_0^5 2t^2 + 1dt \ = \left[\frac{2}{3}t^3 + t\right]_0^5 = \frac{250}{3} + 5 = \frac{265}{3} \ (1\%)$$

(b) $\mathbf{r}'(4) =< 8, 32, 1 >$, $\mathbf{r}''(4) =< 2, 16, 0 >$

$\mathbf{r}'(4) \times \mathbf{r}''(4) =< -16, 2, 64 >$, $|\mathbf{r}'(4)| = 33$

$$|\mathbf{r}'(4) \times \mathbf{r}''(4)| = 2\sqrt{64 + 1 + 1024} = 2\sqrt{1089} = 2\sqrt{3^2 \cdot 11^2} = 66$$

$$\begin{aligned}\kappa(4) &= \frac{|\mathbf{r}'(4) \times \mathbf{r}''(4)|}{|\mathbf{r}'(4)|^3} \ (3\%)\\ &= \frac{66}{33^3} \ (2\%)\\ &= \frac{2}{1089} \ (1\%)\end{aligned}$$

or

$$\begin{aligned}\mathbf{T}(t) &= \frac{1}{2t^2+1} < 2t, 2t^2, 1 > \ (1\%)\\ \mathbf{T}'(t) &= \frac{2}{(2t^2+1)^2} < 1 - 2t^2, 2t, -2t > \ (3\%)\\ \mathbf{T}'(4) &= \frac{2}{33^2} < -31, 8, -8 >\\ |\mathbf{T}'(4)| &= \frac{2}{33}\\ \kappa(4) &= \frac{|\mathbf{T}'(4)|}{|\mathbf{r}'(4)|} \ (2\%)\\ &= \frac{2}{33^2} = \frac{2}{1089} \ (1\%)\end{aligned}$$

6. (15%) The plane $x + y + 2z = 2$ intersects the paraboloid $z = x^2 + y^2$ in an ellipse. Find the points on this ellipse that are nearest to and farthest from the origin.

Sol:

Method1: Lagrange Multiplier

$$f(x, y, z) = x^2 + y^2 + z^2 \quad (1)$$

$$g_1(x, y, z) = x + y + 2z \quad (2)$$

$$g_2(x, y, z) = x^2 + y^2 - z \quad (3)$$

f is to measure distance from the origin. g_1 and g_2 are restrictions.

To find out critical points of f with these restrictions, one needs to solve:

$\nabla f = a\nabla g_1 + \nabla g_2$

$= (2x, 2y, 2z) = a(1, 1, 2) + b(2x, 2y, -1)$

Combine with restrictions ,we have 5 unknowns and 5 equations:

$$2x = a + 2xb \tag{4}$$

$$2y = a + 2yb \tag{5}$$

$$2z = 2a - b \tag{6}$$

$$x + y + 2z = 2 \tag{7}$$

$$z = x^2 + y^2 \tag{8}$$

By (4) and (5) , $x = y \rightarrow$ (4) and (5)to be

$$z = \frac{2-2x}{2} = 1 - x \tag{9}$$

$$z = 2x^2 \tag{10}$$

By (9),(10) $x = \frac{1}{2}$ or $x = -1$ Plug back to (4)-(8),we have two solutions

$$(x_1, y_1, z_1, a_1, b_1) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{2}{3}, \frac{1}{3}) \tag{11}$$

$$(x_2, y_2, z_2, a_2, b_2) = (-1, -1, 2, \frac{10}{3}, \frac{8}{3}) \tag{12}$$

Maxia is $\sqrt{(-1)^2 + (-1)^2 + 2^2} = \sqrt{6}$

Minima is $\sqrt{\frac{1}{2}^2 + \frac{1}{2}^2 + \frac{1}{2}^2} = \sqrt{\frac{3}{8}}$

Remark: One can choose $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ and do similar calculations to gain critical points.

Score criterion: Eq(4)-(8) +8 ; Solve x,y,z correctly +2 each ; max/min +1

Method2: Lagrange Multiplier (modified)

Since

$$f(x,y) = x^2 + y^2 + z(x,y)^2 = x^2 + y^2 + (x^2 + y^2)^2$$

and one constrain condition:

$$g(x,y) = x + y + 2z = x + y + 2(x^2 + y^2) = 2$$

then use Lagrange Multiplier

$$\nabla f(x,y) = a\nabla g(x,y)$$

to solve x,y and z

7. (15%) Find and classify the critical points of the function $f(x,y) = (x^2 + y^2)e^{y^2 - x^2}$.

   Sol:

   $f_x(x,y) = 2xe^{y^2-x^2}(1 - x^2 - y^2)$

   $f_y(x,y) = 2ye^{y^2-x^2}(1 + x^2 + y^2)$ (3%)

   $f_x = f_y = 0 \Longrightarrow (x,y) = (\pm 1, 0), (0,0)$ (6 %)

   $f_{xx} = (2 - 10x^2 - 2y^2 + 4x^4 + 4x^2y^2)e^{y^2-x^2}$

   $f_{yy} = (2 + 10y^2 - 2x^2 + 4x^2y^2 + 4y^4)e^{y^2-x^2}$

   $f_{yz} = 4(-x^2 - y^2)e^{y^2-x^2}$

   $(x,y) = (0,0) \Longrightarrow D > 0, f_{xx} > 0$: local minimum

   $(x,y) = (\pm 1, 0) \Longrightarrow D < 0$: saddle points (6 (Additional error points cause some deduction.)

8. (12%) Let $T(x,y,z) = e^{-x^2-3y^2-9z^2}$.

   (a) Find the directional derivative of $T(x,y,z)$ at $P_0 = (2,-1,2)$ toward the point $(3,-3,3)$.

   (b) Find the maximum of directional derivative of $T(x,y,z)$ at $P_0 = (2,-1,2)$.

   Sol:

   (a) $P_0 = (2,-1,2)$, $P_1 = (3,-3,3)$

   $\overrightarrow{P_0P_1} = (1,-2,1)$ (1%)

$$v = \frac{\overrightarrow{P_0P_1}}{\overline{P_0P_1}} = \frac{1}{\sqrt{6}}(1, -2, 1) \text{ (2\%)}$$

$$\nabla T = e^{-x^2-3y^2-9z^2}(-2x, -6y, -18z) \text{ (2\%)}$$

$$\begin{aligned} T_v(2, -1, 2) &= v \bullet \nabla T(2, -1, 2) \text{ (2\%)} \\ &= \frac{-52}{\sqrt{6}} e^{-43} \text{ (1\%)} \end{aligned}$$

(b) maximim : $u // \nabla T$ and $|u| = 1$ (2%)

so take $u = \frac{1}{\sqrt{337}}(-2, 3, -18)$

$T_u = 2e^{-43}\sqrt{337}$ (2%)