982微積分甲01-05班期中考解答與評分標準

1. (12%) (a) Evaluate $\lim\limits_{n\to\infty} n(\sqrt[n]{2}-1)$.

   (b) Find the interval of convergence of the power series $p_1(x)=\sum\limits_{n=1}^{\infty}(\sqrt[n]{2}-1)^n(x-1)^n$.

   (c) Find the interval of convergence of the power series $p_2(x)=\sum\limits_{n=1}^{\infty}(\sqrt[n]{2}-1)(x-1)^n$.

   Sol:

   (a) (2%) $\lim\limits_{n\to\infty} n(2^{\frac{1}{n}}-1)=\lim\limits_{x\to 0^+}\dfrac{2^x-1}{x}=f'(0)=\ln 2$, where $f(x)=2^x$.

   (b) (3%) $\dfrac{1}{R}=\lim\limits_{n\to\infty}\sqrt[n]{|\,(\sqrt[n]{2}-1)(x-1)\,|^n}=|\,x-1\,|\lim\limits_{n\to\infty}|\,\sqrt[n]{2}-1\,|=0 \;\Rightarrow\; R=\infty$

   the convergence set is $\mathbb{R}$

   (c) (7%) radius of convergence (3%)

   $\lim\limits_{n\to\infty}\sqrt[n]{|\,\sqrt[n]{2}-1\,||\,x-1\,|^n}=|\,x-1\,|\lim\limits_{n\to\infty}(\sqrt[n]{2}-1)^{\frac{1}{n}}$, and

   $\lim\limits_{n\to\infty}\left(\sqrt[n]{2}-1\right)^{\frac{1}{n}}=\exp\left(\lim\limits_{n\to\infty}\dfrac{\ln(\sqrt[n]{2}-1)}{n}\right)=\exp\left(\lim\limits_{n\to\infty}\dfrac{\ln[n(\sqrt[n]{2}-1)]-\ln n}{n}\right)=e^0=1.$

   Thus radius is 1.

   At $x=2$ (2%): By part (a) and limit comparison test, since $\lim\limits_{n\to\infty}\dfrac{2^{\frac{1}{n}}-1}{\frac{1}{n}}=\ln 2$ and $\sum\limits_{n}\dfrac{1}{n}=\infty$,

   $p_2(2)=\sum\limits_{n=1}^{\infty}(\sqrt[n]{2}-1)$ is divergent.

   At $x=0$ (2%): Let $a_n=2^{\frac{1}{n}}-1$, clearly a_n is decreasing to 0. By alternating series test, $p_2(0)=\sum\limits_{n=1}^{\infty}(-1)^n a_n$ is convergent.

   The convergence interval is $[0,2)$.

2. (9%) Let $f(x)=\arctan\dfrac{4-x^2}{4+x^2}$, $x\in\mathbb{R}$.

   (a) Evaluate $f'(x)$.

   (b) Find the power series representation for $f(x)$ about $x=0$. That is, find $c_0,\ c_1,\ c_2,\cdots$ such that $f(x)=c_0+\sum\limits_{n=1}^{\infty}c_nx^n$. What is the radius of convergence of this power series?

Sol:

$f(x) = \arctan\dfrac{4-x^2}{4+x^2} \Rightarrow f'(x) = \dfrac{1}{1+(\frac{4-x^2}{4+x^2})^2}\cdot\dfrac{-2x(4+x^2)-2x(4-x^2)}{(4+x^2)^2} = \dfrac{-8x}{x^4+16}$ (3%)

$C_0 = f(0) = \dfrac{\pi}{4}$ (1%)

$$\begin{aligned} f(x) &= C_0 + \int_0^x -\frac{t}{2}\cdot\frac{1}{1+(\frac{t}{2})^4}dt \\ &= C_0 + \int_0^x -\frac{t}{2}\cdot\sum_{n=0}^{\infty}(-1)^n\cdot(\frac{t}{2})^{4n}dt \\ &= C_0 + \int_0^x \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2^{4n+1}}t^{4n+1}dt = C_0 + \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2^{4n+1}(4n+2)}x^{4n+2} \text{ (2\%)} \end{aligned}$$

$C_n = \dfrac{(-1)^{\frac{n+2}{4}}}{2^{n-1}n}$ if $n = 2 \mod 4$, $C_n = 0$ otherwise (2%). Radius of convergence is 2 (1%).

3. (8%) Let $g(x) = \dfrac{1}{2x^2-3x+1}$. Find $g^{(10)}(-1)$, the tenth derivative of $g(x)$ at $x=-1$.

Sol:

(Method 1.)

$g(x) = \dfrac{1}{2x^2-3x+1} = \dfrac{1}{(2x-1)(x-1)} = \dfrac{1}{x-1} - \dfrac{2}{2x-1}$ (3%)

$g^{(10)}(x) = (-1)^{10}\times 10!\times\dfrac{1}{(x-1)^{11}} - 2^{10}\times(-1)^{10}\times 10!\times\dfrac{2}{(2x-1)^{11}}$

$g^{(10)}(-1) = \dfrac{-10!}{2^{11}} + (\dfrac{2}{3})^{11}\times 10! = 10!((\dfrac{2}{3})^{11} - \dfrac{1}{2^{11}})$ (5%)

(Method 2.)

4. (12%) Consider the power series $H(x) = x - \left(1+\frac{1}{2}\right)x^2 + \left(1+\frac{1}{2}+\frac{1}{3}\right)x^3 - \left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)x^4 + \cdots = \sum_{k=1}^{\infty}\left(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{k}\right)(-1)^{k-1}x^k$.

(a) Find the interval of convergence of $H(x)$.

(b) Express $(1+x)H(x)$ as a power series about $x=0$. Recognize it as the Maclaurin series of certain function $Q(x)$.

(c) Evaluate $\frac{1}{2} + \left(1+\frac{1}{2}\right)\left(\frac{1}{2^2}\right) + \left(1+\frac{1}{2}+\frac{1}{3}\right)\left(\frac{1}{2^3}\right) + \cdots + \left(1+\frac{1}{2}+\cdots+\frac{1}{k}\right)\left(\frac{1}{2^k}\right) + \cdots = m$.

Sol:

(a) (3%) Let $a_k = \frac{1}{k}$. Since $\lim_{k\to\infty} |\frac{a_k}{a_{k+1}}| = 1 - 0 = 1$, then the radius is 1. Besides, at $x = \pm 1$, the $k - th$ term of $H(x)$ satisfies $\lim_{k\to\infty} |a_k| = \sum_{k=1}^{\infty} \frac{1}{k} = \infty$.

By the test for divergence, $H(1)$ and $H(-1)$ are divergent.

Thus the interval of convergence is $(-1, 1)$.

(b) (5%) Since $(1+x)H(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(x)^k}{k}$ (2%)

$Q(x) = \ln(1+x)$ by $-\ln(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k}$, $x \in (-1, 1)$. (3%)

(c) (4%) Take $x = \frac{-1}{2}$. Then we have $m = -H(\frac{-1}{2}) = \frac{-Q(\frac{-1}{2})}{1 - 2\frac{1}{2}} = -2 \cdot \ln(\frac{-1}{2}) = \ln 4$

5. (15%) Let $\mathbf{r}(t) = \langle \frac{\cos \pi t}{\pi}, \frac{\sin \pi t}{\pi}, \frac{4}{3} t^{\frac{3}{2}} \rangle$, $t \geq 0$.

(a) Find the length of the arc $0 \leq t \leq 2$ of $\mathbf{r}(t)$.

(b) Find $\mathbf{T}(2)$, the unit tangent vector when $t = 2$.

(c) Find $\kappa(2)$, the curvature when $t = 2$.

(d) Find $\mathbf{N}(2)$ and $\mathbf{B}(2)$, the principal unit normal vector and the binormal vector when $t = 2$, respectively.

Sol:

(a) Since $\mathbf{r}'(t) =< -\sin \pi t, \cos \pi t, 2t^{\frac{1}{2}} >$ and $|\mathbf{r}'(t)| = \sqrt{4t+1}$,

$$s = \int_0^2 \sqrt{4t+1}dt = \frac{1}{6}(4t+1)^{\frac{3}{2}}\Big|_0^2 = \frac{13}{3} \text{ (3\%)}$$

(b) $\mathbf{T}(2) = \frac{\mathbf{r}'(2)}{|\mathbf{r}'(2)|} = (0, \frac{1}{3}, \frac{2\sqrt{2}}{3})$ (3%)

(c) $$\mathbf{T}'(t) = \frac{< -\pi \cos \pi t, -\pi \sin \pi t, t^{-\frac{1}{2}} > \sqrt{4t+1} - \frac{2< -\sin \pi t, \cos \pi t, 2t^{1/2} >}{\sqrt{4t+1}}}{4t+1}$$

$$\mathbf{T}'(2) = \frac{1}{9}\Big\{ < -\pi, 0, \frac{1}{\sqrt{2}} > \cdot 3 - \frac{2}{3} < 0, 1, 2\sqrt{2} > \Big\} = \frac{1}{9} < -3\pi, -\frac{2}{3}, \frac{3\sqrt{2}}{2} - \frac{4\sqrt{2}}{3} >$$

$$|\mathbf{T}'(2)| = \frac{1}{9}\Big(9\pi^2 + \frac{4}{9} + \frac{1}{18}\Big)^{\frac{1}{2}}$$

$$\kappa(2) = \frac{|\mathbf{T}'(2)|}{|\mathbf{r}'(2)|} = \frac{\sqrt{36\pi^2 + 2}}{54} \text{ (3\%)}$$

(d) $\mathbf{N}(2) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \dfrac{18}{\sqrt{36\pi^2+2}}(-\dfrac{\pi}{2}, -\dfrac{2}{27}, \dfrac{\sqrt{2}}{54})$ (3%)

$\mathbf{B}(2) = \mathbf{T}(2) \times \mathbf{N}(2) = \dfrac{1}{\sqrt{36\pi^2+2}}(\sqrt{2}, -4\pi\sqrt{2}, 2\pi)$ (3%)

6. (12%) Evaluate the following limits.

(a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2} = A$, (b) $\lim\limits_{(x,y)\to(0,0)} \left(\dfrac{xy}{x^2+y^2}\right)^{\frac{1}{x^2}} = B$, (c) $\lim\limits_{(x,y)\to(0,1)} (1+x)^{\frac{x-y}{x}} = C$.

Sol:

(a) A doesn't exist

Along the line $y = x$, $\lim\limits_{x=y\to 0} \dfrac{xy}{x^2+y^2} = \dfrac{1}{2}$.

Along either the line $y = 0(x \neq 0)$ or $x = 0(y \neq 0)$, $\dfrac{xy}{x^2+y^2} = 0$ (using polar coordinate to explain is ok).

So the limit does not exist. (4 pts)

(b) Both answers (1) and (2) are acceptable (4pts), but you have to explain the reasons.

Answer(1). The question is problematic because $(\dfrac{xy}{x^2+y^2})^{\frac{1}{x^2}}$ is not defined when $xy < 0$, and such (x, y) is present in any neighborhood of $(0, 0)$.

Answer(2). $B = 0$

solution: It follows from $0 \leq| xy |\leq \dfrac{1}{2}(x^2 + y^2)$, then $0 \leq (\dfrac{| xy |}{x^2+y^2})^{\frac{1}{x^2}} \leq (\dfrac{1}{2})^{\frac{1}{x^2}}$.

Since $\lim\limits_{(x,y)\to(0,0)} (\dfrac{1}{2})^{\frac{1}{x^2}} = 0$, B exists and equals 0 by Squeeze theorem.

(c) Both answers (1) and (2) are acceptable (4pts), but you have to explain the reasons.

Answer(1). The question is also problematic because $(1 + x)^{\frac{x-y}{x}}$ is not defined in the neighborhood of $(0, 1)$ when $x = 0$.

Answer(2). C=e^{-1}

solution:

$$\begin{aligned}
\lim_{(x,y)\to(0,1)} (1+x)^{\frac{x-y}{x}} &= \lim_{(x,y)\to(0,1)} (1+x)(1+x)^{\frac{-y}{x}} \\
&= \lim_{(x,y)\to(0,1)} (1+x) \lim_{(x,y)\to(0,1)} (1+x)^{\frac{-y}{x}} = \lim_{(x,y)\to(0,1)} \exp(\frac{-y}{x}\ln(1+x)) \\
&= \exp(\lim_{(x,y)\to(0,1)} -y\frac{\ln(1+x)}{x}) \text{ (by the continuity of exponential function)} \\
&= \exp(\lim_{(x,y)\to(0,1)} (-y) \lim_{(x,y)\to(0,1)} \frac{\ln(1+x)}{x}) \text{ (both limits exist)} \\
&= \exp(-1\times 1) = e^{-1} \text{ (4 pts)}
\end{aligned}$$

Many people pass $y=1$ into the limit of $f(x,y)$ first and then find $\lim_{x\to 0} f(x,-1)$ or set $y = mx+1$ to find the limit or e.c.t.

But these are wrong ways. In general, $\lim_{(x,y)\to(a,b)} f(x,y) \neq \lim_{x\to a}\lim_{y\to b} f(x,y)$.

7. (8%) Find the tangent plane at $(1,e,e^2)$ on the surface $x+\ln(y^2)+\ln(z^4)=11$.

Sol:

Set $f(x,y,z) = x+\ln(y^2)+\ln(z^4)-11$, $\bigtriangledown f(x,y,z) = (1,\frac{2}{y},\frac{4}{z})$, $\bigtriangledown f(1,e,e^2) = (1,\frac{2}{e},\frac{4}{e^2})$ (3 pts)

The tangent plane is $(x-1)+\frac{2}{e}(y-e)+\frac{4}{e^2}(z-e^2)=0$ (5 pts)

8. (12%) Suppose that $f(x,y)$ is differentiable at $(1,0)$, $f(1,0)=3$, and $D_{\mathbf{u}_0}f(1,0) = \frac{3}{\sqrt{2}}$ in the direction $\mathbf{u}_0 = (\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$, while $D_{\mathbf{v}_0}f(1,0) = \frac{-1}{\sqrt{2}}$ in the direction $\mathbf{v}_0 = (\frac{-1}{\sqrt{2}},\frac{1}{\sqrt{2}})$.

(a) Find the maximum value of the directional derivative $D_{\mathbf{u}}f$ at $(x,y)=(1,0)$.

(b) Let the space curve $\mathbf{r}(t)$ be the intersection of the surfaces $z=f(x,y)$ and $z=x+y^2+2$. Find the parametric equation for the tangent line to $\mathbf{r}(t)$ at the point $(x,y,z)=(1,0,3)$.

Sol:

(a) Let $\nabla f(1,0) = (a,b)$, then we have $\frac{1}{\sqrt{2}}a+\frac{1}{\sqrt{2}}b = \frac{3}{\sqrt{2}}$ and $\frac{-1}{\sqrt{2}}a+\frac{1}{\sqrt{2}}b = \frac{-1}{\sqrt{2}}$.
So $\nabla f(1,0) = (a,b) = (2,1)$.
This implies the maximum of the directional derivatives is $|\nabla f(1,0)| = |(2,1)| = \sqrt{5}$, which occurs when the direction is $\left(\frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}}\right)$.

(b) Let $F(x,y,z) = f(x,y) - z$ and $G(x,y,z) = x + y^2 + 2 - z$.

$\nabla F|_{(1,0,3)} = (2,1,-1)$, $\nabla G|_{(1,0,3)} = (1,2y,-1)\,|_{(1,0,3)} = (1,0,-1)$

$\nabla F|_{(1,0,3)} \times \nabla G|_{(1,0,3)} = (2,1,-1) \times (1,0,-1) = (-1,1,-1)$

So the tangent line is $\dfrac{x-1}{-1} = \dfrac{y-0}{1} = \dfrac{z-3}{-1}$.

評分標準:

(a) (1) 列出方程組: 3分 (2) 方程組解出答案: 2分 (3) 寫出方向導數最大值爲何: 1分 (4) 其餘錯誤酌量給分

(b) (1) 算出兩個區面的梯度函數: 各2分 (2) 求出切線方向向量: 1分 (3) 寫出切線方程式: 1分 (4) 其餘錯誤酌量給分

9. (12%) Let $U = x^3 y$, and x, y, and t satisfy

$$\begin{cases} x^5 + y = t, \\ x^2 + y^3 = t^2. \end{cases} \qquad (*)$$

Around $t = -1$ and $(x,y) = (-1,0)$, by Implicit Function Theorem, the relationship $(*)$ defines two differentiable functions $x = x(t)$ and $y = y(t)$.

(a) Evaluate $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$ at $t = -1$ and $(x,y) = (-1,0)$. (Hint. Differentiate $(*)$ and derive a system of equations of $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$.)

(b) Evaluate $\dfrac{dU}{dt}$ at $t = -1$ and $(x,y) = (-1,0)$.

Sol:

(a) Differentiating (*) with respect to t, we have

$$5x^4 \frac{dx}{dt} + \frac{dy}{dt} = 1, \qquad (3\text{ 分})$$

$$2x\frac{dx}{dt} + 3y^2 \frac{dy}{dt} = 2t. \qquad (3\text{ 分})$$

Thus,

$$\frac{dx}{dt} = \frac{3y^2 - 2t}{15x^4y^2 - 2x}\bigg|_{t=-1,(x,y)=(-1,0)} = \frac{-1}{-1} = 1,$$
$$\frac{dy}{dt} = \frac{10x^4t - 2x}{15x^4y^2 - 2x}\bigg|_{t=-1,(x,y)=(-1,0)} = \frac{-10+2}{2} = -4.$$

(若前面做對，則到此做對可累計各得 4 分，此小題滿分 8 分)

Or, at $t = -1, (x, y) = (-1, 0)$,

$$5\frac{dx}{dt} + \frac{dy}{dt} = 1, \qquad (3\text{ 分})$$
$$-2\frac{dx}{dt} = -2. \qquad (3\text{ 分})$$

Hence $\frac{dx}{dy} = 1, \frac{dy}{dt} = -4$. (若前面做對，則到此做對可累計各得 4 分，此小題滿分 8 分)

(b)

$$U = x^3y,\ \frac{dU}{dt} = \frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt}, \qquad (\text{得 2 分})$$
$$= 3x^2y\frac{dx}{dt} + x^3\frac{dy}{dt}. \qquad (\text{到此做對，累計得 3 分})$$

At $t = -1, (x, y) = (-1, 0)$, $\frac{dU}{dt} = (-1)^3 \cdot (-4) = 4$. (到此做對，累計得 4 分)