982微積分乙01-05班期末考解答與評分標準

1. (10%) (有移民的人口模型) 令 ρ 爲移民率 (移民數/單位時間) 爲一常數。設 $P(t)$ 表示在時間 t 之人口總數，滿足 $P(t_0) = P_0$ 及 $P'(t) = \lambda P(t) + \rho$，其中 $\lambda > 0$ 爲常數。求 $P(t)$ 的一般式。

   Sol:

   $$\int_{t_0}^{t} \frac{P'(t)}{P(t) + \frac{\rho}{\lambda}}\, dt = \int_{t_0}^{t} \lambda\, dt$$

   $$\ln \frac{P(t) + \frac{\rho}{\lambda}}{P_0 + \frac{\rho}{\lambda}} = \lambda(t - t_0)$$

   $$P(t) = -\frac{\rho}{\lambda} + (P_0 + \frac{\rho}{\lambda})e^{\lambda(t-t_0)}$$

   Write down the exact solution: 10 pts.

   Write down the general solution: 8 pts.

   Write down or derive the formula: 5 pts.

   Others : give you some points accorddng to the wrong extent.

2. (10%) 解微分方程 $\frac{dy}{dt} = 2ty + 4t$，$y(0) = 1$。

   Sol:

   $$e^{\int -2tdt} = e^{-t^2} \quad (6 \text{ pts})$$

   $$e^{-t^2}y' - 2tye^{-t^2} = 4te^{-t^2}$$

   $$\Rightarrow e^{-t^2}y = \int 4te^{-t^2}dt = -2e^{-t^2} + C$$

   $$\Rightarrow y = Ce^{t^2} - 2 \quad (3 \text{ pts})$$

   $$y(0) = 1 \Rightarrow 1 = C - 2 \Rightarrow C = 3 \Rightarrow y(t) = 3e^{t^2} - 2 \quad (1 \text{ pt})$$

3. (10%) 解微分方程 $\frac{dy}{dt} = (y-1)(y-2)$，$y(0) = 0$。

Sol:

$$
\begin{aligned}
&\frac{dy}{dt} = (y-1)(y-2) \\
&\Rightarrow \int (\frac{1}{(y-1)(y-2)})dy = \int dt \ \ \text{(3 pts)} \\
&\Rightarrow \int (\frac{1}{y-2} - \frac{1}{y-1})dy = \int dt \ \ \text{(2 pts)} \\
&\Rightarrow \log(|y-2|) - \log(|y-1|) = t + c \\
&\Rightarrow \log(\left|\frac{y-2}{y-1}\right|) = t + c \\
&\Rightarrow \left|\frac{y-2}{y-1}\right| = e^{t+c} \\
&\Rightarrow y = \frac{2-c_1e^t}{1-c_1e^t} \ \ \text{(3 pts for any one of above forms)} \\
&\Rightarrow y = \frac{2-2e^t}{1-2e^t} \ \ \text{(Since } y(0)=0\text{)} \ \ \text{(2 pts)}
\end{aligned}
$$

4. (10%) 令 V 爲一隨機變數。而 $V = k$，$k \geq 2$，表示在連續進行白努利試驗時，到第 k 次才出現兩次 + 事件。計算 $\mathrm{P}(V = k)$。(設 p 爲一次白努利試驗出現 + 之機率，而 $q = 1 - p$。)

Sol:

$$P(V=k) = C_1^{k-1}pq^{k-2} \cdot p = C_1^{k-1}p^2q^{k-2} \ \ \text{(10 pts)}$$

Any other answers similar to the correct one receives at most 4 points.

You will get 4 points if your answer has one place different to the correct one.

You will get 2 points if your answer has two places different to the correct one.

You will get 0 point if your answer has more than two places different to the correct one.

5. (20%) 設 X，Y 爲隨機變數取值 1 或 2。已知

$\mathrm{P}(X=1, Y=1) = \frac{4}{19}$，$\mathrm{P}(X=1, Y=2) = \frac{2}{19}$，$\mathrm{P}(X=2, Y=1) = \frac{7}{19}$，$\mathrm{P}(X=2, Y=2) = \frac{6}{19}$

求 $\mathrm{P}(X=1)$，$\mathrm{P}(X=2)$，$\mathrm{E}(X)$ 及 $\mathrm{Var}(X)$。(各5%)

Sol:

(a)

$$\begin{aligned}
\mathrm{P}(X=1) &= \mathrm{P}(X=1,Y=1)+\mathrm{P}(X=1,Y=2) \quad \text{(1 pt)}\\
&= \frac{4}{19}+\frac{2}{19} \quad \text{(2 pts)}\\
&= \frac{6}{19} \quad \text{(2 pts)}
\end{aligned}$$

(b)

$$\begin{aligned}
\mathrm{P}(X=2) &= \mathrm{P}(X=2,Y=1)+\mathrm{P}(X=2,Y=2) \quad \text{(1 pt)}\\
&= \frac{7}{19}+\frac{6}{19} \quad \text{(2 pts)}\\
&= \frac{13}{19} \quad \text{(2 pts)}
\end{aligned}$$

(c)

$$\begin{aligned}
\mathrm{E}(X) &= 1\cdot\mathrm{P}(X=1)+2\cdot\mathrm{P}(X=2) \quad \text{(1 pt)}\\
&= \frac{6}{19}+2\cdot\frac{13}{19} \quad \text{(2 pts)}\\
&= \frac{32}{19} \quad \text{(2 pts)}
\end{aligned}$$

(d) $\mathrm{Var}(X)=\mathrm{E}(X^2)-\mathrm{E}(X)^2$ (1 pt)

$\mathrm{E}(X^2)=1^2\cdot\frac{6}{19}+2^2\cdot\frac{13}{19}=\frac{58}{19}$ (2 pts)

$\mathrm{Var}(X)=\frac{58}{19}-(\frac{32}{19})^2=\frac{78}{361}$ (2 pts)

6. (15%) 設隨機變數 X 之機率密度函數爲

$$f(t)=\begin{cases} ct(1-t) & \text{若 } 0\le t\le 1\\ 0 & \text{其他}\end{cases}$$

其中 c 爲常數。求 c 之値，再求 $\mathrm{E}(X)$，$\mathrm{Var}(X)$。(各5%)

Sol:

Since $f(x)$ is a random function,

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 cx(1-x)\,dx \quad \text{(2 pts)}\\
&= c\int_0^1 x-x^2\,dx = c\left(\frac{x^2}{2}-\frac{x^3}{3}\right)_0^1 = c\left(\frac{1}{2}-\frac{1}{3}\right)=\frac{c}{6}
\end{aligned}$$

Hence $c = 6$ (3 pts)

$$
\begin{aligned}
\mathrm{E}(x) &= \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x(6x(1-x))\,dx \quad \text{(2 pts)}\\
&= 6\int_0^1 x^2 - x^3\,dx = 6\left(\frac{x^3}{3} - \frac{x^4}{4}\right)_0^1\\
&= 6\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{2} \quad \text{(3 pts)}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{V}(x)\left(= \int_{-\infty}^{\infty} (x - E(x))^2 f(x)\,dx\right) &= E(x^2) - (E(x))^2 \quad \text{(2 pts)}\\
&= \int_{-\infty}^{\infty} x^2 f(x)\,dx - \left(\frac{1}{2}\right)^2 = 6\int_0^1 x^3 - x^4\,dx - \left(\frac{1}{2}\right)^2\\
&= 6\left(\frac{x^4}{4} - \frac{x^5}{5}\right)_0^1 - \left(\frac{1}{2}\right)^2 = 6\left(\frac{1}{4} - \frac{1}{5}\right) - \left(\frac{1}{2}\right)^2 = \frac{1}{20} \quad \text{(3 pts)}
\end{aligned}
$$

7. (15%) 若 X , Y 爲獨立之隨機變數並且有指數型機率密度函數

$$f_X(t) = \lambda e^{-\lambda t} = f_Y(t) \text{ 對 } t \geq 0 \text{ , 而 } f_X(t) = 0 = f_Y(t) \text{ 對 } t < 0 \text{ 。}$$

求 $Z = X + Y$ 之機率密度函數 $f_Z(t)$ 及 $\mathrm{E}(Z)$, $\mathrm{Var}(Z)$ 。(各5%)

Sol:

(a) $f_Z(t) = \int_{-\infty}^{\infty} f_X(x) f_Y(t-x)\,dx$ (2 pts)

The region of integration is $0 \leq x \leq t$

Since $0 \leq y \leq \infty$, $-\infty \leq -y \leq 0$, so $0 \leq x = t - y \leq t$

$f_Z(t) = \int_0^t \lambda e^{-\lambda x} \lambda e^{-\lambda(t-x)}\,dx = \int_0^t \lambda^2 e^{-\lambda t} dx = \lambda^2 t e^{-\lambda t}$ (3 pts)

(b) X, Y have the same distribution, so

$$
\begin{aligned}
\mathrm{E}(Z) = \mathrm{E}(X) + \mathrm{E}(Y) = 2\mathrm{E}(X) &= 2\int_0^{\infty} \lambda t e^{-\lambda t}\,dt\\
&= \lim_{b\to\infty} 2\int_0^b \lambda t e^{-\lambda t}\,dt \quad \text{(2 pts)}\\
&= \lim_{b\to\infty} -2te^{-\lambda t}\Big|_0^b + 2\lim_{b\to\infty}\int_0^b e^{-\lambda t} dt = \frac{2}{\lambda} \quad \text{(3 pts)}
\end{aligned}
$$

(c) Since $X,\ Y$ are i.i.d, $\mathrm{Cov}(X,Y)=0$ and

$$\begin{aligned}\mathrm{Var}(Z) &= \mathrm{Var}(X)+\mathrm{Var}(Y)+2\mathrm{Cov}(X,Y)\\ &= \mathrm{Var}(X)+\mathrm{Var}(Y)=2\mathrm{Var}(X)\\ &= 2[\mathrm{E}(X^2)-(\mathrm{E}(X))^2] \quad \text{(2 pts)}\end{aligned}$$

$$\begin{aligned}\mathrm{E}(X^2) &= \int_0^\infty t^2\lambda e^{-\lambda t}\,dt = \lim_{b\to\infty}\int_0^b t^2\lambda e^{-\lambda t}\,dt\\ &= \lim_{b\to\infty} -e^{-\lambda t}t^2\Big|_0^b + 2\lim_{b\to\infty}\int_0^b te^{-\lambda t} = \frac{2}{\lambda^2}\end{aligned}$$

So $\mathrm{Var}(X)=\dfrac{1}{\lambda^2}$, and $\mathrm{Var}(Z)=\dfrac{2}{\lambda^2}$ (3 pts)

8. (10%) 某公司之電話通數平均每小時 30 通。求在 6 分鐘內至少有兩通電話之機率。(假設此隨機現象遵守 Poisson 過程。)

Sol:

Let X be the random variable of the times of phone calls during 6 seconds. Observe that the Poisson density is

$$f_X(x)=\begin{cases}\dfrac{\lambda^x e^{-\lambda}}{x!} & x=0,1,2,\ldots\\ 0 & \text{otherwise}\end{cases} \qquad \text{(3 pts)}$$

and $\lambda = 6\cdot\dfrac{30}{60}=3$ (3 pts). Hence, we have

$$\mathrm{P}(X\geq 2)=1-f_X(0)-f_X(1)=1-e^{-3}-3e^{-3}=1-4e^{-3} \quad \text{(4 pts)}$$