982微積分甲08-13班期末考解答與評分標準

1. (8%) Evaluate $\int_0^4 \int_{\frac{y}{2}}^2 e^{x^2}\,dx\,dy$.

   Sol:

$$\begin{aligned}
&\int_0^4 \int_{\frac{y}{2}}^2 e^{x^2}\,dxdy \\
&= \int\int_D e^{x^2}\,dA \\
&= \int_0^2 \int_0^{2x} e^{x^2}\,dydx \quad \text{(4 pts)} \\
&= \int_0^2 2xe^{x^2}\,dx \quad \text{(2 pts)} \\
&= e^{x^2}\Big|_0^2 \\
&= e^4 - 1 \quad \text{(2 pts)}
\end{aligned}$$

2. (10%) Find the area of the surface $z = \frac{2}{3}(x^{\frac{3}{2}} + y^{\frac{3}{2}}),\ 0 \le x,\ y \le 1$.

   Sol:

   Parametrization:

$$\begin{aligned}
x &= \left(u, v, \frac{2}{3}\left(u^{\frac{3}{2}} + v^{\frac{3}{2}}\right)\right) \quad \text{and} \quad u, v \in [0,1] \\
x_u &= \left(1, 0, u^{\frac{1}{2}}\right) \\
x_v &= \left(0, 1, v^{\frac{1}{2}}\right) \quad \text{(3 pts)} \\
(x_u \times x_v) &= \left(-u^{\frac{1}{2}}, -v^{\frac{1}{2}}, 1\right) \\
|x_u \times x_v| &= \sqrt{1 + u + v}
\end{aligned}$$

$$\begin{aligned}
\text{Area of surface} &= \int_0^1 \int_0^1 \sqrt{1+u+v}\; du\; dv \quad \text{(4 pts)} \\
&= \frac{2}{3}\int_0^1 \left[(1+u+v)^{\frac{3}{2}}\right]_0^1 \; dv \\
&= \frac{2}{3}\int_0^1 (2+v)^{\frac{3}{2}} - (1+v)^{\frac{3}{2}}\; dv \\
&= \frac{4}{15}\left[(2+v)^{\frac{5}{2}} - (1+v)^{\frac{5}{2}}\right]_0^1 \\
&= \frac{4}{15}\left[3^{\frac{5}{2}} - 2^{\frac{5}{2}} - 2^{\frac{5}{2}} + 1\right] \quad \text{(3 pts)}
\end{aligned}$$

Calculation errors: -1 pt.

Minor mistakes: -1 pt.

3. (10%) Evaluate $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{4-x^2-y^2} (x^2+y^2)\, dz\, dy\, dx.$

   Sol:

$$\begin{aligned}
&\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{4-x^2-y^2} (x^2+y^2)\, dzdydx \\
&= \int_0^{2\pi}\int_0^2\int_0^{4-r^2} r^2\cdot r\, dzdrd\theta \quad (x = r\cos\theta, y = r\sin\theta) \quad \text{(5 pts)} \\
&= \frac{32\pi}{3} \quad \text{(5 pts)}
\end{aligned}$$

4. (10%) Evaluate $\displaystyle\iint_{x^2+xy+y^2\le 1} e^{-(x^2+xy+y^2)}\, dx\, dy.$

   Sol:

$$\begin{aligned}
&\iint_{x^2+xy+y^2\le 1} e^{-(x^2+xy+y^2)} dxdy \\
&= \iint_{(x+y/2)^2+(\sqrt{3}y/2)^2\le 1} e^{-((x+y/2)^2+(\sqrt{3}y/2)^2)} dxdy \\
&= \iint_{u^2+v^2\le 1} e^{u^2+v^2} \left| \begin{vmatrix} 1 & -1/\sqrt{3} \\ 0 & 2/\sqrt{3} \end{vmatrix} \right| dudv && \text{(4 pts)} \\
&= \frac{2}{\sqrt{3}}\int_0^{2\pi}\int_0^1 e^{-r^2} rdrd\theta && \text{(前面做對, 到此累計得 7 pts)} \\
&= \frac{2\pi(1-e^{-1})}{\sqrt{3}} && \text{(前面做對, 到此累計得 10 pts)}
\end{aligned}$$

5. (16%) Let $\mathbf{F}(x,y,z) = y^2\mathbf{i} + (2xy + e^{3z})\mathbf{j} + 3ye^{3z}\mathbf{k}$

(a) Find the potential function of $\mathbf{F}$.

(b) Compute the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $C : \mathbf{r}(t) = \langle 4t, 3\cos t, 3\sin t\rangle,\ 0 \le t \le \frac{\pi}{2}$.

Sol:

(a) $F(x, y, z) = (2xy, 2xy + e^{3z}, 3ye^{3z})$.

Assume f is the potential function (Real value function such that $\nabla f = F$),

then $f_x = 2xy$ imply $f(x, y, z) = xy^2 + h(y, z)$.

Then take derivative with y variable on both side we got $f_y = 2xy + h_y(y, z)$.

By definition of F we also have, $h_y(y, z) = e^{3z} \Rightarrow h(y, z) = ye^{3z} + g(z)$.

Since $f_z = 3ye^{3z}$, this imply $f_z = h_z = 3ye^{3z} + g_z(z) = 3ye^{3z}$, i.e, $g_z = 0$.

So $f = xy^2 + ye^{3z} + C$ for some constant.

(b)

$$\begin{aligned}\int_C F dr &= \int_C \nabla f dr = f(r(\pi/2)) - f(r(0)) \\ &= f(2\pi, 0, 3) - f(0, 3, 0) = 2\pi \times 0^2 + 3 \times 0 \times e^0 - (0 \times 3^2 + 3e^0) = -3\end{aligned}$$

評分標準:

(a) If the answer is right then you got 8 point.

If not, you can gain 2 at each step while you are trying to solve h and g.

(b) $\int_C F dr = \int_C \nabla f dr = f(r(\pi/2)) - f(r(0))$ This step cost 4 point.

$= f(2\pi, 0, 3) - f(0, 3, 0)$ This step cost 2 point.

$= 2\pi \times 0^2 + 3 \times 0 \times e^0 - (0 \times 3^2 + 3e^0) = -3$. Answer cost 2 point.

6. (16%) Let $P = \dfrac{y^3}{(x^2+y^2)^2}$, $Q = -\dfrac{xy^2}{(x^2+y^2)^2}$. Let C be the counterclockwise ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$, and D be the region inside C but outside the unit circle $x^2 + y^2 = 1$.

(a) Evaluate $\displaystyle\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$.

(b) Evaluate $\displaystyle\int_C P\,dx + Q\,dy$.

Sol:

(a) $\dfrac{\partial Q}{\partial x} = \dfrac{3x^2y^2 - y^4}{(x^2+y^2)^3}$ (2 pts), $\dfrac{\partial P}{\partial y} = \dfrac{3x^2y^2 - y^4}{(x^2+y^2)^3}$ (2 pts)

so $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 0$

$\displaystyle\iint_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dA = 0$ (2 pts)

(b) Let E be the counterclockwise unite circle. Then $\partial D = C \cup E$

By Green theorem, $\displaystyle\int\int_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dA = \int_C Pdx + Qdy + \int_{-E} Pdx + Qdy$

So $\displaystyle\int_C Pdx + Qdy = \int_E Pdx + Qdy$ (4 pts)

To compute $\displaystyle\int_E Pdx + Qdy$, parametrize $E = \{(\cos\theta, \sin\theta) | 0 \le \theta \le 2\pi\}\}$

Then

$$\begin{aligned}
\int_C Pdx + Qdy = \int_E Pdx + Qdy &= \int_0^{2\pi} (\sin\theta)^3 d\cos\theta - \cos\theta(\sin\theta)^2 d\sin\theta \qquad \text{(2 pts)} \\
&= \int_0^{2\pi} [-(\sin\theta)^4 - (\cos\theta)^2(\sin\theta)^2] d\theta \\
&= \int_0^{2\pi} [-(\sin\theta)^2] d\theta = \int_0^{2\pi} \left[-\frac{1-\cos 2\theta}{2} \right] d\theta \qquad \text{(2 pts)} \\
&= -\frac{\theta}{2} + \frac{\cos 2\theta}{4} \Big|_0^{2\pi} = -\pi \qquad \text{(2 pts)}
\end{aligned}$$

7. (10%) Evaluate $\displaystyle\iint_S \frac{x^2}{\sqrt{1+x^2+y^2}} dS$, where S is the helicoid with equation $\mathbf{r}(u,v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + v\mathbf{k},\ 0 \le u \le 1,\ 0 \le v \le 2\pi.$

Sol:

$$\gamma_u = (\cos v\ , \sin v\ , 0\ )$$

$$\gamma_v = (-u\sin v\ , u\cos v\ , 1\ )$$

$$\gamma_u \times \gamma_v = (\sin v\ , -\cos v\ , u) \quad \text{(3 pts)}$$

$$|\ \gamma_u \times \gamma_v\ | = \sqrt{u^2+1} \quad \text{(2pts)}$$

We have

$$\iint_S \frac{x^2}{\sqrt{1+x^2+y^2}} dS = \int_0^{2\pi} \int_0^1 \frac{u^2\cos^2 v}{\sqrt{1+u^2}} \sqrt{1+u^2}\, du\, dv \quad (2 \text{ pts})$$
$$= \int_0^{2\pi} \int_0^1 u^2 \cos^2 v\, du\, dv$$
$$= \frac{\pi}{3} \quad (3 \text{ pts})$$

8. (10%) Compute the integral $\iint\limits_S \text{curl}\mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x,y,z) = (e^{z^2}+y)\mathbf{i} + (4z-y)\mathbf{j} + (8x\sin y)\mathbf{k}$ and S is the part of $z = 4 - x^2 - y^2$ above the xy-plane with orientation given by the upward unit normal vector.

   Sol:

Solution 1: By Stokes' Theorem

$$\iint_S \text{curl}\mathbf{F} \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} \quad (2 \text{ pts})$$

The boundary of S is the circle $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + 0\mathbf{k}$. (2 pts)

And

$$\begin{aligned}
\mathbf{r}'(t) &= (-2\sin t\mathbf{i} + 2\cos t\mathbf{j} + 0\mathbf{k}) \quad (1 \text{ pt}) \\
&= \oint_{\partial S} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \quad (2 \text{ pts}) \\
&= \int_0^{2\pi} ((e^{0^2} + 2\sin t)\mathbf{i} + (4\times 0 - 2\sin t)\mathbf{j} \\
&+ (8\cos t \sin(\sin t))\mathbf{k}) \cdot (-2\sin t\mathbf{i} + 2\cos t\mathbf{j} + 0\mathbf{k})\, dt \quad (1 \text{ pt}) \\
&= \int_0^{2\pi} -2\sin t - 4\sin^2 t - 4\sin t\cos t\, dt \quad (1 \text{ pt}) \\
&= \int_0^{2\pi} -2\sin t - 2 + 2\cos 2t - 2\sin 2t\, dt \\
&= (2\cos t - 2 + 2\sin 2t + 2\cos 2t)\Big|_0^{2\pi} \\
&= -4\pi \quad (1 \text{ pt})
\end{aligned}$$

Solution 2: $\text{curl}\mathbf{F} = (8x\cos y - 4)\mathbf{i} + (2ze^{z^2} - 8\sin y)\mathbf{j} - \mathbf{k}$ (2 pts)

The surface of S is $\mathbf{S}(x,y) = (x, y, 4 - x^2 - y^2)$ (1 pt)

$\mathbf{S}_x = (1,0,-2x), \mathbf{S}_y = (0,1,-2y), \mathbf{S}_x \times \mathbf{S}_y = (-2x, 2y, 1)$ (2 pts)

$$\iint_S \text{curl}\mathbf{F} \cdot d\mathbf{S} = \iint_D (8x\cos y - 4)\mathbf{i} + (2ze^{z^2} - 8\sin y)\mathbf{j} - \mathbf{k} \cdot (-2x, 2y, 1)\, dA \quad (2 \text{ pts})$$

Because of the symmetric $-\iint_S dA = -4\pi$ (3 pts)

Solution 3: $\text{curl}\mathbf{F} = (8x\cos y - 4)\mathbf{i} + (2ze^{z^2} - 8\sin y)\mathbf{j} - \mathbf{k}$ (2 pts)

Let D be the disk $0 = 4 - x^2 - y^2$ where the orientation given $(0,0,-1)$

$$\iint_S \text{curl}\mathbf{F} \cdot d\mathbf{S} = \iiint_V div(\text{curl}\mathbf{F})dV - \iint_D \text{curl}\mathbf{F} \cdot d\mathbf{A} \quad (4 \text{ pts})$$

$$= 0 - \iint_D d\mathbf{A} = -4\pi \quad (3 \text{ pts})$$

(1 pt for knowing $\iiint_V div(\text{curl}\mathbf{F})dV = 0$)

Solution 4: By Stokes' Theorem

$$\iint_S \text{curl}\mathbf{F} \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} \quad (2 \text{ pts})$$

$\text{curl}\mathbf{F} = (8x\cos y - 4)\mathbf{i} + (2ze^{z^2} - 8\sin y)\mathbf{j} - \mathbf{k}$ (2 pts)

Let D be the disk $0 = 4 - x^2 - y^2$ where the orientation given $(0,0,1)$

By Stokes' Theorem

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_D \text{curl}\mathbf{F} \cdot d\mathbf{A} \quad (3 \text{ pts})$$

$$= -\iint_D dA = -4\pi \quad (3 \text{ pts})$$

9. (10%) Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x,y,z) = 2y^2\mathbf{j} - z\mathbf{k}$ and S is the surface of the solid enclosed by $y = x^2 + z^2$ and $y = 1$ with outward normal vector.

Sol:

method 1: Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{j} = 0\mathbf{i} + 2y^2\mathbf{j} - z\mathbf{k}$, then $div\mathbf{F} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z} = 4y - 1$ (2 pts)

By the divergence theorem, we have $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_R div\mathbf{F}\, dV$ (3 pts) where R is the region bounded by S. Hence

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^1 \int_{r^2}^1 (4y-1)r\, dy dr d\theta \quad \text{(3 pts)}$$

$$= 2\pi \int_0^1 r - 2r^5 + r^3 dr = \frac{5\pi}{6} \quad \text{(2 pts)}$$

method 2:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

$$= \iint_{S_1} \mathbf{F} \cdot \mathbf{n}\, dS + \iint_{S_2} \mathbf{F} \cdot \mathbf{n}\, dS \quad \text{(2 pts)}$$

where $\mathbf{n}$ is the outer normal of S, $S_1 = S \cap \{y = x^2 + z^2\}$, and $S_2 = S \cap \{y = 1\}$.

$$\iint_{S_1} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_A (0, 2(x^2+z^2)^2, -z) \cdot (2x, -1, 2z)\, dA \quad \text{(3 pts)}$$

$$= \int_0^{2\pi} \int_0^1 (-2r^4 - 2r^2 \sin^2\theta) r\, dr d\theta$$

$$= -\frac{7\pi}{6} \quad \text{(3 pts)}$$

$$\iint_{S_2} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_A (0, 2, -z) \cdot (0, 1, 0)\, dA$$

$$= \int_0^{2\pi} \int_0^1 2r dr d\theta = 2\pi \quad \text{(2 pts)}$$