98學年度第1學期 微積分乙期中考解答

1. (10%) 令 $f(x) = x^{\ln x}$, 求 $f'(x)$ 。

   Sol:

   $f(x) = x^{\ln x} = e^{(\ln x)^2}$

   $f'(x) = e^{(\ln x)^2} \cdot ((\ln x)^2)' = x^{\ln x} \cdot 2\ln x \cdot \frac{1}{x} = 2(\ln x)x^{\ln x - 1}$

2. (10%) 令 $f(x) = \ln(x + \sqrt{1+x^2})$, 求 $f'(x)$ 。

   Sol:

   $f'(x) = \frac{1}{x+\sqrt{1+x^2}}(1 + \frac{x}{\sqrt{1+x^2}}) = \frac{1}{\sqrt{1+x^2}}$

3. (10%) 求 $f(x) = \frac{1}{1+\tan x}$ 在 $x = \frac{\pi}{4}$ 處之切線 。

   Sol:

   Let $f(x) = \frac{1}{(1+\tan x)}$, then $f'(x) = -\frac{\sec^2 x}{(1+\tan x)^2}$.

   Because $f(\frac{\pi}{4}) = \frac{1}{2}$, $\quad f'(\frac{\pi}{4}) = -\frac{1}{2}$, we get $y - \frac{1}{2} = -\frac{1}{2}(x - \frac{\pi}{4})$.

4. (10%) 令 $f(x) = \frac{1}{x}$, $1 \le x \le 2$ 。求 $\xi \in (1,2)$ 使得 $f'(\xi) = \frac{f(2)-f(1)}{2-1}$ 。

   Sol:

   $f(x) = \frac{1}{x} \Rightarrow f'(x) = \frac{-1}{x^2}$

   $\frac{f(2)-f(1)}{2-1} = -\frac{1}{2}$

   $\Rightarrow \xi^2 = 2 \Rightarrow \xi = \sqrt{2}$

5. (10%) 用線性逼近求 $\tan^{-1}(1.02)$ 之近似值 。(用弳度量)

   Sol:

   Let $f(x) = \arctan x \Rightarrow f'(x) = \frac{1}{x^2+1}$. We have

$$\begin{aligned} f(1.02) &\approx f(1) + (1.02-1)\cdot f'(1) \\ &= \frac{\pi}{4} + 0.02 \cdot \frac{1}{1^2+1} \\ &= \frac{\pi}{4} + 0.01 \end{aligned}$$

6. (15%) 求曲線 $x^3+y^3-3x^2y=3$ 在點 $(1,2)$ 之 y' 及 y''。(前者10%，後者5%)

Sol:

Differentiating both sides of $x^3+y^3-3x^2y=3$ with respect to x and regarding y as a function of x, we have

$$3x^2+3y^2y'-6xy-3x^2y'=0. \qquad (*)$$

Substituting $x=1$ and $y=2$ into $(*)$, we get $y'=1$.

To find y'', we differentiate $(*)$ implicitly with respect to x to obtain

$$6x+6y(y')^2+3y^2y''-6y-6xy'-6xy'-3x^2y''=0.$$

Plugging $x=1$, $y=2$ and $y'=1$ into the above equation, we get $y''=\dfrac{2}{3}$.

7. (10%) 令 $f(x)=\sqrt{1+x+x^2}$，$x\geq 0$。求 $(f^{-1})'(\sqrt{3})=?$ (建議：不要直接求出反函數 $f^{-1}(x)$ 之表示式。)

Sol:

Since $x\geqslant 0$, $f^{-1}(x)$ exist. This will give:

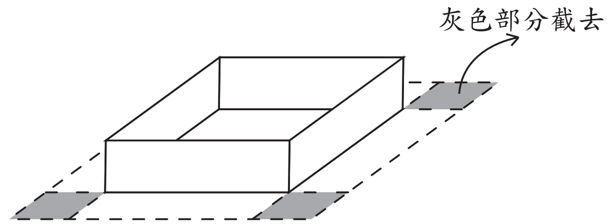
$$f(1)=3\Rightarrow f^{-1}(\sqrt{3})=1$$

We have: $(f^{-1})'(x)=\dfrac{1}{f'(f^{-1}(x))}$ and $f'(x)=\dfrac{2x+1}{2\sqrt{1+x+x^2}}$

Hence,

$$\begin{aligned}(f^{-1})'(\sqrt{3}) &= \frac{1}{f'(f^{-1}(\sqrt{3}))}\\ &= \frac{1}{f'(1)}\\ &= \frac{1}{\frac{3}{2\sqrt{3}}}\\ &= \frac{2\sqrt{3}}{3}\quad (\text{or } \frac{2}{\sqrt{3}})\end{aligned}$$

8. (10%) 邊長爲 12 公分之正方形紙板 。四個角處 , 各截去大小相同之小正方形 。剩餘部分折成無蓋之紙盒 。如圖 。求最大容積 。(不必測試所得值是否極大 。)



Sol:

Let the height of the box be x. Then both two lengthes of the box is $12-2x$. The volume of the box is

$$f(x)=x(12-2x)^2,\ 0\le x\le 6.$$

Since $f(x)$ is differential everywhere,

the maximal value of $f(x)$ can only happen at $\{x|f'(x)=0, or\ x=0,6\}$.

$$\begin{aligned} f'(x) &= x2(12-2x)(-2)+(12-2x)^2(1) \\ &= (12-2x)(-4x+12-2x)=12(6-x)(2-x) \end{aligned}$$

$f(2)=128, f(6)=0, f(0)=0$

Maximum value is $f(2)=128(cm^3)$

9. (15%) 令 $f(x)=x^5-5x+1$ 。回答下列問題 , 寫出計算過程 , 將答案填入空格 。

(a) 求出 $f(x)$ 之極大發生在 $x=$ __________ , 極小發生在 $x=$ __________ 。

(b) 求出 $f(x)$ 遞增之區間爲 __________ , 遞減之區間爲 __________ 。

(c) 求出 $f(x)$ 之反曲點在 $x=$ __________ 。

(d) 求出 $f(x)$ 凹向上之區間爲 __________ , 凹向下之區間爲 __________ 。

(e) 繪出 $y=f(x)$ 之圖 。

Sol:

$f(x)=x^5-5x+1$

$\Rightarrow f'(x) = 5x^4 - 5 = 5(x^4 - 1) = 5(x^2 + 1)(x^2 - 1)$

$\Rightarrow f'(x) = 0 \Longleftrightarrow x = \pm 1$

When $x < -1$, $f'(x) > 0$,

when $-1 < x < 1$, $f'(x) < 0$,

when $x > 1$, $f'(x) > 0$.

(a) The relative maximal point is $x = -1$, the relative minimal point is $x = 1$.

(b) The increasing area is $x < -1$ or $x > 1$, the decreasing area is $-1 < x < 1$.

Since $f''(x) = 20x^3$, $\Rightarrow f''(x) = 0 \Longleftrightarrow x = 0$.

When $x < 0$, $f''(x) < 0$; when $x > 0$, $f''(x) > 0$.

(c) The inflective point is $x = 0$.

(d) Concave upward area is $x > 0$, concave downward area is $x < 0$,

and $f(-1) = -1 + 5 + 1 = 5$, $f(1) = 1 - 5 + 1 = -3$, $f(0) = 0 - 0 + 1 = 1$.

(e) The graph is