Section 9.1 Modeling with Differential Equations

21. A population is modeled by the differential equation

$$\frac{dP}{dt} = 1.2P\left(1 - \frac{P}{4200}\right)$$

(a) For what values of P is the population increasing?

(b) For what values of P is the population decreasing?

(c) What are the equilibrium solutions?

Solution:

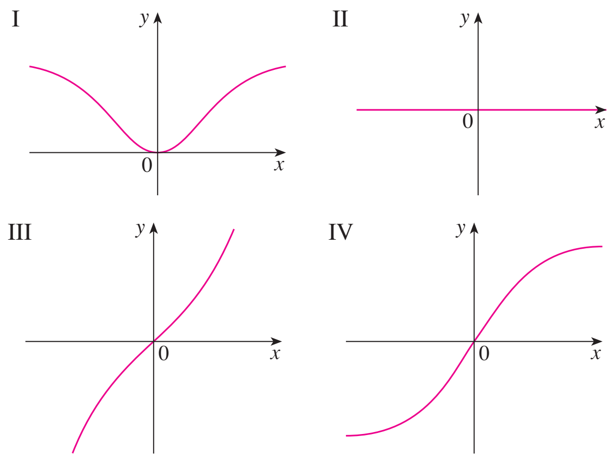
(a) $\frac{dP}{dt} = 1.2P\left(1 - \frac{P}{4200}\right)$. Now $\frac{dP}{dt} > 0 \;\Rightarrow\; 1 - \frac{P}{4200} > 0$ [assuming that $P > 0$] $\Rightarrow \frac{P}{4200} < 1 \;\Rightarrow$ $P < 4200 \;\Rightarrow$ the population is increasing for $0 < P < 4200$.

(b) $\frac{dP}{dt} < 0 \;\Rightarrow\; P > 4200$

(c) $\frac{dP}{dt} = 0 \;\Rightarrow\; P = 4200$ or $P = 0$

25. Match the differential equations with the solution graphs labeled I–IV. Give reasons for your choices.

(a) $y' = 1 + x^2 + y^2$ (b) $y' = xe^{-x^2-y^2}$ (c) $y' = \frac{1}{1+e^{x^2+y^2}}$ (d) $y' = \sin(xy)\cos(xy)$



Solution:

(a) $y' = 1 + x^2 + y^2 \geq 1$ and $y' \to \infty$ as $x \to \infty$. The only curve satisfying these conditions is labeled III.

(b) $y' = xe^{-x^2-y^2} > 0$ if $x > 0$ and $y' < 0$ if $x < 0$. The only curve with negative tangent slopes when $x < 0$ and positive tangent slopes when $x > 0$ is labeled I.

(c) $y' = \frac{1}{1 + e^{x^2+y^2}} > 0$ and $y' \to 0$ as $x \to \infty$. The only curve satisfying these conditions is labeled IV.

(d) $y' = \sin(xy)\,\cos(xy) = 0$ if $y = 0$, which is the solution graph labeled II.

29. Differential equations have been used extensively in the study of drug dissolution for patients given oral medications. One such equation is the Weibull equation for the concentration $c(t)$ of the drug:

$$\frac{dc}{dt} = \frac{k}{t^b}(c_s - c)$$

where k and c_s are positive constants and $0 < b < 1$. Verify that

$$c(t) = c_s(1 - e^{-\alpha t^{1-b}})$$

is a solution of the Weibull equation for $t > 0$, where $\alpha = k/(1-b)$. What does the differential equation say about how drug dissolution occurs?

Solution:

If $c(t) = c_s\left(1 - e^{-\alpha t^{1-b}}\right) = c_s - c_s e^{-\alpha t^{1-b}}$ for $t > 0$, where $k > 0$, $c_s > 0$, $0 < b < 1$, and $\alpha = k/(1-b)$, then

$\frac{dc}{dt} = c_s\left[0 - e^{-\alpha t^{1-b}} \cdot \frac{d}{dt}(-\alpha t^{1-b})\right] = -c_s e^{-\alpha t^{1-b}} \cdot (-\alpha)(1-b)t^{-b} = \frac{\alpha(1-b)}{t^b} c_s e^{-\alpha t^{1-b}} = \frac{k}{t^b}(c_s - c)$. The equation for c indicates that as t increases, c approaches c_s. The differential equation indicates that as t increases, the rate of increase of c decreases steadily and approaches 0 as c approaches c_s.