Section 7.5 Strategy for Integration

8. Three integrals are given that, although they look similar, may require different techniques of integration. Evaluate the integrals.

(a) $\int e^x\sqrt{e^x-1}dx$ (b) $\int \frac{e^x}{\sqrt{1-e^{2x}}}dx$ (c) $\int \frac{1}{\sqrt{e^x-1}}dx$

Solution:

(a) Let $u = e^x - 1$, so that $du = e^x\,dx$. Thus, $\int e^x\sqrt{e^x-1}\,dx = \int u^{1/2}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(e^x-1)^{3/2} + C$.

(b) Let $u = e^x$, so that $du = e^x\,dx$. Thus, $\displaystyle\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1}u + C = \sin^{-1}(e^x) + C$.

(c) Let $u = \sqrt{e^x-1}$, so that $u^2 = e^x - 1 \;\Rightarrow\; 2u\,du = e^x\,dx$, and $\dfrac{2u\,du}{u^2+1} = dx$. Then

$$\int \frac{1}{\sqrt{e^x-1}}\,dx = \int \frac{1}{u}\frac{2u\,du}{u^2+1} = 2\int \frac{1}{u^2+1}\,du = 2\tan^{-1}u + C = 2\tan^{-1}\sqrt{e^x-1} + C.$$

27. Evaluate the integral. $\int e^{x+e^x}dx$.

Solution:

Let $u = e^x$. Then $\int e^{x+e^x}\,dx = \int e^{e^x}e^x\,dx = \int e^u\,du = e^u + C = e^{e^x} + C$.

44. Evaluate the integral. $\displaystyle\int \frac{1+\sin x}{1+\cos x}dx$.

Solution:

$$\int \frac{1+\sin x}{1+\cos x}\,dx = \int \frac{(1+\sin x)(1-\cos x)}{(1+\cos x)(1-\cos x)}\,dx = \int \frac{1-\cos x+\sin x-\sin x\,\cos x}{\sin^2 x}\,dx$$

$$= \int \left(\csc^2 x - \frac{\cos x}{\sin^2 x} + \csc x - \frac{\cos x}{\sin x}\right)dx$$

$$\overset{s}{=} -\cot x + \frac{1}{\sin x} + \ln|\csc x - \cot x| - \ln|\sin x| + C$$

The answer can be written as $\dfrac{1-\cos x}{\sin x} - \ln(1+\cos x) + C$.

76. Evaluate the integral. $\displaystyle\int \frac{x^2}{x^6+3x^3+2}dx$.

Solution:

$$\int \frac{x^2}{x^6+3x^3+2}\,dx = \int \frac{x^2\,dx}{(x^3+1)(x^3+2)} = \int \frac{\frac{1}{3}\,du}{(u+1)(u+2)} \quad \begin{bmatrix} u = x^3, \\ du = 3x^2\,dx \end{bmatrix}.$$

Now $\dfrac{1}{(u+1)(u+2)} = \dfrac{A}{u+1} + \dfrac{B}{u+2} \;\Rightarrow\; 1 = A(u+2) + B(u+1)$. Setting $u = -2$ gives $B = -1$. Setting $u = -1$ gives $A = 1$. Thus,

$$\frac{1}{3}\int \frac{du}{(u+1)(u+2)} = \frac{1}{3}\int\left(\frac{1}{u+1} - \frac{1}{u+2}\right)du = \frac{1}{3}\ln|u+1| - \frac{1}{3}\ln|u+2| + C$$

$$= \tfrac{1}{3}\ln\left|x^3+1\right| - \tfrac{1}{3}\ln\left|x^3+2\right| + C$$

93. Evaluate the integral $\displaystyle\int_0^{\pi/6} \sqrt{1+\sin 2\theta}d\theta$.

Solution:

$$\int_0^{\pi/6} \sqrt{1+\sin 2\theta}\, d\theta = \int_0^{\pi/6} \sqrt{\left(\sin^2\theta + \cos^2\theta\right) + 2\sin\theta\cos\theta}\, d\theta = \int_0^{\pi/6} \sqrt{(\sin\theta+\cos\theta)^2}\, d\theta$$

$$= \int_0^{\pi/6} |\sin\theta + \cos\theta|\, d\theta = \int_0^{\pi/6} (\sin\theta+\cos\theta)\, d\theta \qquad \left[\begin{array}{c}\text{since integrand is}\\ \text{positive on } [0, \pi/6]\end{array}\right]$$

$$= \Big[-\cos\theta + \sin\theta\Big]_0^{\pi/6} = \left(-\frac{\sqrt{3}}{2} + \frac{1}{2}\right) - (-1+0) = \frac{3-\sqrt{3}}{2}$$

Alternate solution:

$$\int_0^{\pi/6} \sqrt{1+\sin 2\theta}\, d\theta = \int_0^{\pi/6} \sqrt{1+\sin 2\theta} \cdot \frac{\sqrt{1-\sin 2\theta}}{\sqrt{1-\sin 2\theta}}\, d\theta = \int_0^{\pi/6} \frac{\sqrt{1-\sin^2 2\theta}}{\sqrt{1-\sin 2\theta}}\, d\theta$$

$$= \int_0^{\pi/6} \frac{\sqrt{\cos^2 2\theta}}{\sqrt{1-\sin 2\theta}}\, d\theta = \int_0^{\pi/6} \frac{|\cos 2\theta|}{\sqrt{1-\sin 2\theta}}\, d\theta = \int_0^{\pi/6} \frac{\cos 2\theta}{\sqrt{1-\sin 2\theta}}\, d\theta$$

$$= -\frac{1}{2}\int_1^{1-\sqrt{3}/2} u^{-1/2}\, du \qquad [u = 1-\sin 2\theta,\, du = -2\cos 2\theta\, d\theta]$$

$$= -\tfrac{1}{2}\Big[2u^{1/2}\Big]_1^{1-\sqrt{3}/2} = 1 - \sqrt{1-(\sqrt{3}/2)}$$