Section 7.1 Integration by Parts

2. Evaluate the integral using integration by parts with the indicated choices of u and dv.

$$\int \sqrt{x}\ln x dx; \quad u = \ln x, \ dv = \sqrt{x}dx$$

Solution:

Let $u = \ln x$, $dv = \sqrt{x}\,dx \quad \Rightarrow \quad du = \dfrac{1}{x}\,dx$, $v = \frac{2}{3}x^{3/2}$. Then by Equation 2,

$$\int \sqrt{x}\ln x\,dx = \tfrac{2}{3}x^{3/2}\ln x - \int \tfrac{2}{3}x^{3/2}\cdot\frac{1}{x}\,dx = \tfrac{2}{3}x^{3/2}\ln x - \int \tfrac{2}{3}x^{1/2}\,dx = \tfrac{2}{3}x^{3/2}\ln x - \tfrac{4}{9}x^{3/2} + C.$$

42. Evaluate the integral $\int_0^t e^s \sin(t-s)ds$.

Solution:

Let $u = \sin(t-s)$, $dv = e^s\,ds \quad \Rightarrow \quad du = -\cos(t-s)\,ds$, $v = e^s$. Then

$I = \int_0^t e^s \sin(t-s)\,ds = \Big[e^s\sin(t-s)\Big]_0^t + \int_0^t e^s\cos(t-s)\,ds = e^t\sin 0 - e^0\sin t + I_1$. For I_1, let $U = \cos(t-s)$,

$dV = e^s\,ds \quad \Rightarrow \quad dU = \sin(t-s)\,ds$, $V = e^s$. So $I_1 = \big[e^s\cos(t-s)\big]_0^t - \int_0^t e^s\sin(t-s)\,ds = e^t\cos 0 - e^0\cos t - I$.

Thus, $I = -\sin t + e^t - \cos t - I \quad \Rightarrow \quad 2I = e^t - \cos t - \sin t \quad \Rightarrow \quad I = \frac{1}{2}(e^t - \cos t - \sin t)$.

48. First make a substitution and then use integration by parts to evaluate the integral. $\int \frac{\arcsin(\ln x)}{x}dx$

Solution:

Let $y = \ln x$, so that $dy = \dfrac{1}{x}\,dx$. Thus, $\displaystyle\int \frac{\arcsin(\ln x)}{x}\,dx = \int \arcsin y\,dy$. Now use

parts with $u = \arcsin y$, $dv = dy$, $du = \dfrac{1}{\sqrt{1-y^2}}\,dy$, and $v = y$ to get

$$\int \arcsin y\,dy = y\arcsin y - \int \frac{y}{\sqrt{1-y^2}}\,dy = y\arcsin y + \sqrt{1-y^2} + C = (\ln x)\arcsin(\ln x) + \sqrt{1-(\ln x)^2} + C.$$

60. Use integration by parts to prove the reduction formula.

$$\int \sec^n x dx = \frac{\tan x \sec^{n-2} x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x dx \quad (n \neq 1)$$

Solution:

Let $u = \sec^{n-2} x$, $dv = \sec^2 x\,dx \quad \Rightarrow \quad du = (n-2)\sec^{n-3} x\,\sec x\,\tan x\,dx$, $v = \tan x$. Then, by Equation 2,

$$\begin{aligned}\int \sec^n x\,dx &= \tan x\,\sec^{n-2} x - (n-2)\int \sec^{n-2} x\,\tan^2 x\,dx \\ &= \tan x\,\sec^{n-2} x - (n-2)\int \sec^{n-2} x\,(\sec^2 x - 1)\,dx \\ &= \tan x\,\sec^{n-2} x - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2} x\,dx\end{aligned}$$

so $(n-1)\int \sec^n x\,dx = \tan x\,\sec^{n-2} x + (n-2)\int \sec^{n-2} x\,dx$. If $n-1 \neq 0$, then

$$\int \sec^n x\,dx = \frac{\tan x\,\sec^{n-2} x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx.$$

78. (a) Use integration by parts to show that

$$\int f(x)dx = xf(x) - \int xf'(x)dx$$

(b) If f and g are inverse functions and f' is continuous, prove that

$$\int_a^b f(x)dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} g(y)dy$$

[*Hint*: Use part (a) and make the substitution $y = f(x)$.] (c) In the case where f and g are positive functions and $b > a > 0$, draw a diagram to give a geometric interpretation of part (b).

(d) Use part (b) to evaluate $\int_1^e \ln x dx$.

Solution:

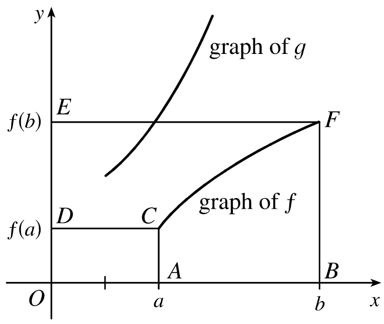
(a) Take $g(x) = x$ and $g'(x) = 1$ in Equation 1.

(b) By part (a), $\int_a^b f(x)\,dx = bf(b) - a\,f(a) - \int_a^b x\,f'(x)\,dx$. Now let $y = f(x)$, so that $x = g(y)$ and $dy = f'(x)\,dx$. Then $\int_a^b x\,f'(x)\,dx = \int_{f(a)}^{f(b)} g(y)\,dy$. The result follows.

(c) Part (b) says that the area of region $ABFC$ is

$= \quad bf(b) \quad - \quad af(a) \quad - \quad \int_{f(a)}^{f(b)} g(y)\,dy$

$=$ (area of rectangle $OBFE$) $-$ (area of rectangle $OACD$) $-$ (area of region $DCFE$)



(d) We have $f(x) = \ln x$, so $f^{-1}(x) = e^x$, and since $g = f^{-1}$, we have $g(y) = e^y$. By part (b),

$$\int_1^e \ln x\,dx = e\ln e - 1\ln 1 - \int_{\ln 1}^{\ln e} e^y\,dy = e - \int_0^1 e^y\,dy = e - \left[\,e^y\,\right]_0^1 = e - (e-1) = 1.$$