Section 6.1 Areas Between Curves

18. Sketch the region enclosed by the given curves. Decide whether to integrate with respect to x or y. Draw a typical approximating rectangle and label its height and width. Then find the area of the region.

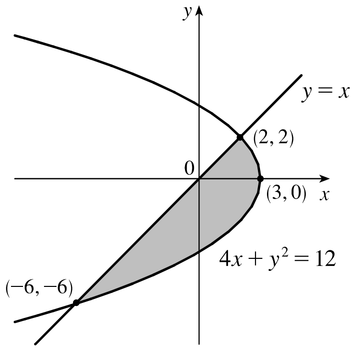
$$4x + y^2 = 12, \quad x = y$$

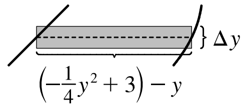
Solution:

$4x + x^2 = 12 \quad \Leftrightarrow \quad (x+6)(x-2) = 0 \quad \Leftrightarrow$

$x = -6$ or $x = 2$, so $y = -6$ or $y = 2$ and

$$A = \int_{-6}^{2} \left[\left(-\tfrac{1}{4}y^2 + 3\right) - y\right] dy$$
$$= \left[-\tfrac{1}{12}y^3 - \tfrac{1}{2}y^2 + 3y\right]_{-6}^{2}$$
$$= \left(-\tfrac{2}{3} - 2 + 6\right) - (18 - 18 - 18)$$
$$= 22 - \tfrac{2}{3} = \tfrac{64}{3}$$





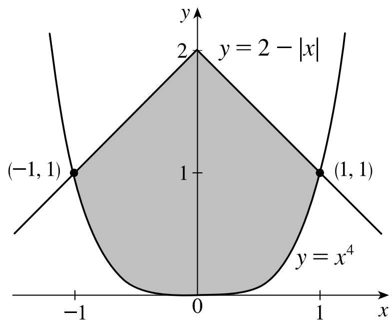
31. Sketch the region enclosed by the given curves and find its area.

$$y = x^4, \quad y = 2 - |x|$$

Solution:

By inspection, we see that the curves intersect at $x = \pm 1$ and that the area of the region enclosed by the curves is twice the area enclosed in the first quadrant.

$$A = 2\int_{0}^{1} \left[(2 - x) - x^4\right] dx = 2\left[2x - \tfrac{1}{2}x^2 - \tfrac{1}{5}x^5\right]_0^1$$
$$= 2\left[\left(2 - \tfrac{1}{2} - \tfrac{1}{5}\right) - 0\right] = 2\left(\tfrac{13}{10}\right) = \tfrac{13}{5}$$

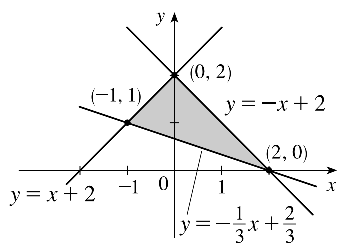


42. Use calculus to find the area of the triangle with the given vertices. $(2, 0)$, $(0, 2)$, $(-1, 1)$.

Solution:

An equation of the line through $(2, 0)$ and $(0, 2)$ is $y = -x + 2$; through $(2, 0)$ and $(-1, 1)$ is $y = -\frac{1}{3}x + \frac{2}{3}$; through $(0, 2)$ and $(-1, 1)$ is $y = x + 2$.

$$A = \int_{-1}^{0} \left[(x+2) - \left(-\tfrac{1}{3}x + \tfrac{2}{3}\right)\right] dx + \int_{0}^{2} \left[(-x+2) - \left(-\tfrac{1}{3}x + \tfrac{2}{3}\right)\right] dx$$
$$= \int_{-1}^{0} \left(\tfrac{4}{3}x + \tfrac{4}{3}\right) dx + \int_{0}^{2} \left(-\tfrac{2}{3}x + \tfrac{4}{3}\right) dx$$
$$= \left[\tfrac{2}{3}x^2 + \tfrac{4}{3}x\right]_{-1}^{0} + \left[-\tfrac{1}{3}x^2 + \tfrac{4}{3}x\right]_0^2$$
$$= 0 - \left(\tfrac{2}{3} - \tfrac{4}{3}\right) + \left(-\tfrac{4}{3} + \tfrac{8}{3}\right) - 0 = 2$$



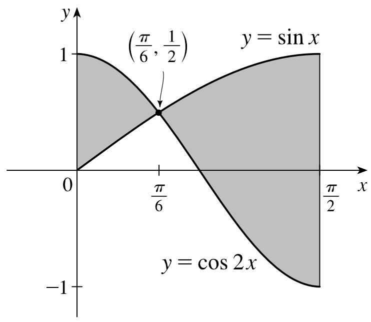
43. Evaluate the integral and interpret it as the area of a region. Sketch the region.

$$\int_0^{\frac{\pi}{2}} |\sin x - \cos 2x| dx$$

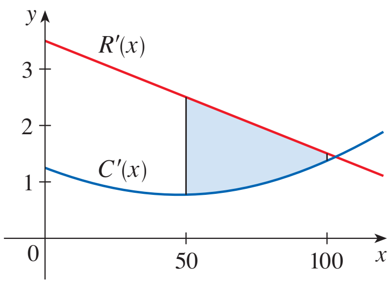
Solution:

The curves intersect when $\sin x = \cos 2x$ (on $[0, \pi/2]$) $\Leftrightarrow$ $\sin x = 1 - 2\sin^2 x$ $\Leftrightarrow$ $2\sin^2 x + \sin x - 1 = 0$ $\Leftrightarrow$ $(2\sin x - 1)(\sin x + 1) = 0$ $\Rightarrow$ $\sin x = \frac{1}{2}$ $\Rightarrow$ $x = \frac{\pi}{6}$.

$$\begin{aligned} A &= \int_0^{\pi/2} |\sin x - \cos 2x|\, dx \\ &= \int_0^{\pi/6} (\cos 2x - \sin x)\, dx + \int_{\pi/6}^{\pi/2} (\sin x - \cos 2x)\, dx \\ &= \left[\tfrac{1}{2}\sin 2x + \cos x\right]_0^{\pi/6} + \left[-\cos x - \tfrac{1}{2}\sin 2x\right]_{\pi/6}^{\pi/2} \\ &= \left(\tfrac{1}{4}\sqrt{3} + \tfrac{1}{2}\sqrt{3}\right) - (0+1) + (0-0) - \left(-\tfrac{1}{2}\sqrt{3} - \tfrac{1}{4}\sqrt{3}\right) \\ &= \tfrac{3}{2}\sqrt{3} - 1 \end{aligned}$$



62. The figure shows graphs of the marginal revenue function R' and the marginal cost function C' for a manufacturer. [Recall from Section 4.7 that $R(x)$ and $C(x)$ represent the revenue and cost when x units are manufactured. Assume that R and C are measured in thousands of dollars.] What is the meaning of the area of the shaded region? Use the Midpoint Rule to estimate the value of this quantity.



Solution:

The area under $R'(x)$ from $x = 50$ to $x = 100$ represents the change in revenue, and the area under $C'(x)$ from $x = 50$ to $x = 100$ represents the change in cost. The shaded region represents the difference between these two values; that is, the increase in profit as the production level increases from 50 units to 100 units. We use the Midpoint Rule with $n = 5$ and $\Delta x = 10$:

$$\begin{aligned} M_5 &= \Delta x\{[R'(55) - C'(55)] + [R'(65) - C'(65)] + [R'(75) - C'(75)] + [R'(85) - C'(85)] + [R'(95) - C'(95)]\} \\ &\approx 10(2.40 - 0.85 + 2.20 - 0.90 + 2.00 - 1.00 + 1.80 - 1.10 + 1.70 - 1.20) \\ &= 10(5.05) = 50.5 \text{ thousand dollars} \end{aligned}$$

Using M_1 would give us $50(2-1) = 50$ thousand dollars.