Section 5.2 The Definite Integral

23. Show that the definite integral is equal to $\lim_{n\to\infty} R_n$ and then evaluate the limit.

$$\int_0^4 (x - x^2)dx, \qquad R_n = \frac{4}{n}\sum_{i=1}^{n}\left[\frac{4i}{n} - \frac{16i^2}{n^2}\right]$$

Solution:

For $\int_0^4 (x - x^2)\,dx$, $\Delta x = \dfrac{4-0}{n} = \dfrac{4}{n}$, and $x_i = 0 + i\,\Delta x = \dfrac{4i}{n}$. Then

$$\int_0^4 (x - x^2)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(\frac{4i}{n}\right)\frac{4}{n} = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(\frac{4i}{n}\right) - \left(\frac{4i}{n}\right)^2\right]\frac{4}{n} = \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n}\left[\frac{4i}{n} - \frac{16i^2}{n^2}\right] = \lim_{n\to\infty} R_n.$$

$$\lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n}\left[\frac{4i}{n} - \frac{16i^2}{n^2}\right] = \lim_{n\to\infty}\frac{4}{n}\left[\frac{4}{n}\sum_{i=1}^{n} i - \frac{16}{n^2}\sum_{i=1}^{n} i^2\right] = \lim_{n\to\infty}\left[\frac{16}{n^2}\frac{n(n+1)}{2} - \frac{64}{n^3}\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty}\left[\frac{8}{n}(n+1) - \frac{32}{3n^2}(n+1)(2n+1)\right]$$

$$= \lim_{n\to\infty}\left[8\left(1+\frac{1}{n}\right) - \frac{32}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right] = 8(1) - \frac{32}{3}(1)(2) = -\frac{40}{3}$$

57. Write as a single integral in the form $\int_a^b f(x)dx$:

$$\int_{-2}^{2} f(x)dx + \int_{2}^{5} f(x)dx - \int_{-2}^{-1} f(x)dx$$

Solution:

$\int_{-2}^{2} f(x)\,dx + \int_{2}^{5} f(x)\,dx - \int_{-2}^{-1} f(x)\,dx = \int_{-2}^{5} f(x)\,dx + \int_{-1}^{-2} f(x)\,dx$ [by Property 5 and reversing limits]

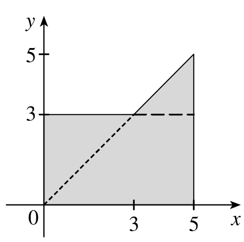
$= \int_{-1}^{5} f(x)\,dx$ [Property 5]

60. Find $\int_0^5 f(x)dx$ if

$$\begin{cases} 3 & \text{for } x < 3 \\ x & \text{for } x \geq 3 \end{cases}$$

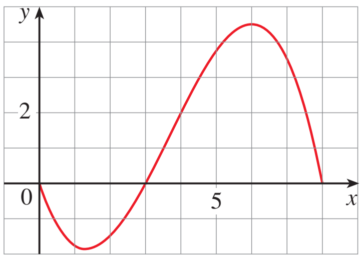
Solution:

If $f(x) = \begin{cases} 3 & \text{for } x < 3 \\ x & \text{for } x \geq 3 \end{cases}$, then $\int_0^5 f(x)\,dx$ can be interpreted as the area of the shaded region, which consists of a 5-by-3 rectangle surmounted by an isosceles right triangle whose legs have length 2. Thus, $\int_0^5 f(x)\,dx = 5(3) + \frac{1}{2}(2)(2) = 17$.



61. For the function f whose graph is shown, list the following quantities in increasing order, from smallest to largest, and explain your reasoning.

(A) $\int_0^8 f(x)dx$ (B) $\int_0^3 f(x)dx$ (C) $\int_3^8 f(x)dx$ (D) $\int_4^8 f(x)dx$ (E) $f'(1)$



Solution:

$\int_0^3 f(x)\,dx$ is clearly less than -1 and has the smallest value. The slope of the tangent line of f at $x=1$, $f'(1)$, has a value between -1 and 0, so it has the next smallest value. The largest value is $\int_3^8 f(x)\,dx$, followed by $\int_4^8 f(x)\,dx$, which has a value about 1 unit less than $\int_3^8 f(x)\,dx$. Still positive, but with a smaller value than $\int_4^8 f(x)\,dx$, is $\int_0^8 f(x)\,dx$. Ordering these quantities from smallest to largest gives us

$$\int_0^3 f(x)\,dx < f'(1) < \int_0^8 f(x)\,dx < \int_4^8 f(x)\,dx < \int_3^8 f(x)\,dx \text{ or } \mathrm{B} < \mathrm{E} < \mathrm{A} < \mathrm{D} < \mathrm{C}$$

68. Use the properties of integrals to verify the inequality without evaluating the integrals.

$$\frac{\pi}{12} \leq \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin x dx \leq \frac{\sqrt{3}\pi}{12}$$

Solution:

If $\frac{\pi}{6} \leq x \leq \frac{\pi}{3}$, then $\frac{1}{2} \leq \sin x \leq \frac{\sqrt{3}}{2}$ ($\sin x$ is increasing on $\left[\frac{\pi}{6}, \frac{\pi}{3}\right]$), so

$\frac{1}{2}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) \leq \int_{\pi/6}^{\pi/3} \sin x\,dx \leq \frac{\sqrt{3}}{2}\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$ [Property 8]; that is, $\frac{\pi}{12} \leq \int_{\pi/6}^{\pi/3} \sin x\,dx \leq \frac{\sqrt{3}\,\pi}{12}$.