Section 4.3 What Derivatives Tell Us about the Shape of a Graph

55. Let $C(x) = x^{1/3}(x+4)$.

(a) Find the intervals of increase or decrease.

(b) Find the local maximum and minimum values.

(c) Find the intervals of concavity and the inflection points.

(d) Use the information from parts (a)–(c) to sketch the graph. You may want to check your work with a graphing calculator or computer.

Solution:

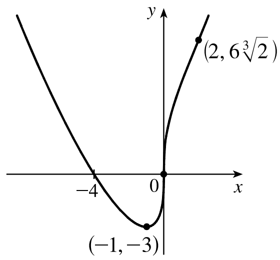
(a) $C(x) = x^{1/3}(x+4) = x^{4/3} + 4x^{1/3} \quad \Rightarrow \quad C'(x) = \frac{4}{3}x^{1/3} + \frac{4}{3}x^{-2/3} = \frac{4}{3}x^{-2/3}(x+1) = \dfrac{4(x+1)}{3\sqrt[3]{x^2}}$. $C'(x) > 0$ if $-1 < x < 0$ or $x > 0$ and $C'(x) < 0$ for $x < -1$, so C is increasing on $(-1, \infty)$ and C is decreasing on $(-\infty, -1)$.

(b) $C(-1) = -3$ is a local minimum value.

(c) $C''(x) = \frac{4}{9}x^{-2/3} - \frac{8}{9}x^{-5/3} = \frac{4}{9}x^{-5/3}(x-2) = \dfrac{4(x-2)}{9\sqrt[3]{x^5}}$.

$C''(x) < 0$ for $0 < x < 2$ and $C''(x) > 0$ for $x < 0$ and $x > 2$, so C is concave downward on $(0, 2)$ and concave upward on $(-\infty, 0)$ and $(2, \infty)$. There are inflection points at $(0, 0)$ and $\left(2, 6\sqrt[3]{2}\right) \approx (2, 7.56)$.

(d)



62. Let $f(x) = \frac{e^x}{1-e^x}$.

(a) Find the vertical and horizontal asymptotes.

(b) Find the intervals of increase or decrease.

(c) Find the local maximum and minimum values.

(d) Find the intervals of concavity and the inflection points.

(e) Use the information from parts (a)–(d) to sketch the graph of f.

Solution:

$f(x) = \dfrac{e^x}{1-e^x}$ has domain $\{x \mid 1 - e^x \neq 0\} = \{x \mid e^x \neq 1\} = \{x \mid x \neq 0\}$.

(a) $\displaystyle\lim_{x\to\infty} \frac{e^x}{1-e^x} = \lim_{x\to\infty} \frac{e^x/e^x}{(1-e^x)/e^x} = \lim_{x\to\infty} \frac{1}{1/e^x - 1} = \frac{1}{0-1} = -1$, so $y = -1$ is a HA.

$\displaystyle\lim_{x\to-\infty} \frac{e^x}{1-e^x} = \frac{0}{1-0} = 0$, so $y = 0$ is a HA. $\displaystyle\lim_{x\to0^+} \frac{e^x}{1-e^x} = -\infty$ and $\displaystyle\lim_{x\to0^-} \frac{e^x}{1-e^x} = \infty$, so $x = 0$ is a VA.

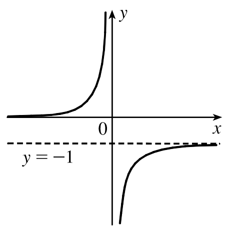
(b) $f'(x) = \dfrac{(1-e^x)e^x - e^x(-e^x)}{(1-e^x)^2} = \dfrac{e^x[(1-e^x)+e^x]}{(1-e^x)^2} = \dfrac{e^x}{(1-e^x)^2}$. $f'(x) > 0$ for $x \neq 0$, so f is increasing on $(-\infty, 0)$ and $(0, \infty)$.

(c) There is no local maximum or minimum.

(d)
$$f''(x) = \frac{(1-e^x)^2 e^x - e^x \cdot 2(1-e^x)(-e^x)}{[(1-e^x)^2]^2} = \frac{(1-e^x)e^x[(1-e^x)+2e^x]}{(1-e^x)^4} = \frac{e^x(e^x+1)}{(1-e^x)^3}$$

$f''(x) > 0 \quad \Leftrightarrow \quad (1-e^x)^3 > 0 \quad \Leftrightarrow \quad e^x < 1 \quad \Leftrightarrow \quad x < 0$ and $f''(x) < 0 \quad \Leftrightarrow \quad x > 0$. So f is CU on $(-\infty, 0)$ and f is CD on $(0, \infty)$. There is no inflection point.

(e)



64. Let $f(x) = x - \frac{1}{6}x^2 - \frac{2}{3}\ln x$.

(a) Find the vertical and horizontal asymptotes.

(b) Find the intervals of increase or decrease.

(c) Find the local maximum and minimum values.

(d) Find the intervals of concavity and the inflection points.

(e) Use the information from parts (a)–(d) to sketch the graph of f.

Solution:

$f(x) = x - \frac{1}{6}x^2 - \frac{2}{3}\ln x$ has domain $(0, \infty)$.

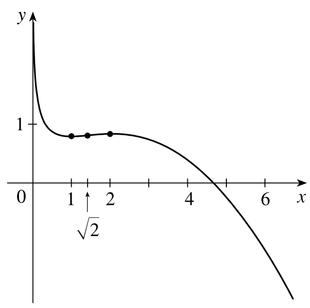
(a) $\lim\limits_{x\to 0^+} \left(x - \frac{1}{6}x^2 - \frac{2}{3}\ln x\right) = \infty$ since $\ln x \to -\infty$ as $x \to 0^+$, so $x = 0$ is a VA. There is no HA.

(b) $f'(x) = 1 - \dfrac{1}{3}x - \dfrac{2}{3x} = \dfrac{3x - x^2 - 2}{3x} = \dfrac{-(x^2 - 3x + 2)}{3x} = \dfrac{-(x-1)(x-2)}{3x}$. $f'(x) > 0 \Leftrightarrow$ $(x-1)(x-2) < 0 \Leftrightarrow 1 < x < 2$ and $f'(x) < 0 \Leftrightarrow 0 < x < 1$ or $x > 2$. So f is increasing on $(1, 2)$ and f is decreasing on $(0, 1)$ and $(2, \infty)$.

(c) f changes from decreasing to increasing at $x = 1$, so $f(1) = \frac{5}{6}$ is a local minimum value. f changes from increasing to decreasing at $x = 2$, so $f(2) = \frac{4}{3} - \frac{2}{3}\ln 2 \approx 0.87$ is a local maximum value.

(d) $f''(x) = -\dfrac{1}{3} + \dfrac{2}{3x^2} = \dfrac{2 - x^2}{3x^2}$. $f''(x) > 0 \Leftrightarrow 0 < x < \sqrt{2}$ and $f''(x) < 0 \Leftrightarrow x > \sqrt{2}$. So f is CU on $\left(0, \sqrt{2}\right)$ and f is CD on $\left(\sqrt{2}, \infty\right)$. There is an inflection point at $\left(\sqrt{2}, \sqrt{2} - \frac{1}{3} - \frac{1}{3}\ln 2\right)$.

(e)



84. For what values of the numbers a and b does the function

$$f(x) = axe^{bx^2}$$

have the maximum value $f(2) = 1$?

Solution:

$f(x) = axe^{bx^2} \Rightarrow f'(x) = a\left[xe^{bx^2} \cdot 2bx + e^{bx^2} \cdot 1\right] = ae^{bx^2}(2bx^2 + 1)$. For $f(2) = 1$ to be a maximum value, we must have $f'(2) = 0$. $f(2) = 1 \Rightarrow 1 = 2ae^{4b}$ and $f'(2) = 0 \Rightarrow 0 = (8b + 1)ae^{4b}$. So $8b + 1 = 0$ $[a \neq 0] \Rightarrow$ $b = -\frac{1}{8}$ and now $1 = 2ae^{-1/2} \Rightarrow a = \sqrt{e}/2$.

95. Show that the function $g(x) = x|x|$ has an inflection point at $(0, 0)$ but $g''(0)$ does not exist.

Solution:

Using the fact that $|x| = \sqrt{x^2}$, we have that $g(x) = x\,|x| = x\sqrt{x^2} \Rightarrow g'(x) = \sqrt{x^2} + \sqrt{x^2} = 2\sqrt{x^2} = 2\,|x| \Rightarrow$ $g''(x) = 2x\left(x^2\right)^{-1/2} = \dfrac{2x}{|x|} < 0$ for $x < 0$ and $g''(x) > 0$ for $x > 0$, so $(0, 0)$ is an inflection point. But $g''(0)$ does not exist.

99. The three cases in the First Derivative Test cover the situations commonly encountered but do not exhaust all possibilities. Consider the functions f, g, and h whose values at 0 are all 0 and, for $x \neq 0$,

$$f(x) = x^4 \sin\frac{1}{x} \qquad g(x) = x^4\left(2 + \sin\frac{1}{x}\right) \qquad h(x) = x^4\left(-2 + \sin\frac{1}{x}\right)$$

(a) Show that 0 is a critical number of all three functions but their derivatives change sign infinitely often on both sides of 0.

(b) Show that f has neither a local maximum nor a local minimum at 0, g has a local minimum, and h has a local maximum.

Solution:

(a) $f(x) = x^4 \sin\frac{1}{x} \quad \Rightarrow \quad f'(x) = x^4 \cos\frac{1}{x}\left(-\frac{1}{x^2}\right) + \sin\frac{1}{x}\,(4x^3) = 4x^3 \sin\frac{1}{x} - x^2 \cos\frac{1}{x}.$

$g(x) = x^4\left(2 + \sin\frac{1}{x}\right) = 2x^4 + f(x) \quad \Rightarrow \quad g'(x) = 8x^3 + f'(x).$

$h(x) = x^4\left(-2 + \sin\frac{1}{x}\right) = -2x^4 + f(x) \quad \Rightarrow \quad h'(x) = -8x^3 + f'(x).$

It is given that $f(0) = 0$, so $f'(0) = \lim\limits_{x\to 0} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x\to 0} \dfrac{x^4 \sin\frac{1}{x} - 0}{x} = \lim\limits_{x\to 0} x^3 \sin\frac{1}{x}$. Since $-\left|x^3\right| \le x^3 \sin\frac{1}{x} \le \left|x^3\right|$ and $\lim\limits_{x\to 0} \left|x^3\right| = 0$, we see that $f'(0) = 0$ by the Squeeze Theorem. Also, $g'(0) = 8(0)^3 + f'(0) = 0$ and $h'(0) = -8(0)^3 + f'(0) = 0$, so 0 is a critical number of f, g, and h.

For $x_{2n} = \dfrac{1}{2n\pi}$ [n a nonzero integer], $\sin\dfrac{1}{x_{2n}} = \sin 2n\pi = 0$ and $\cos\dfrac{1}{x_{2n}} = \cos 2n\pi = 1$, so $f'(x_{2n}) = -x_{2n}^2 < 0$.

For $x_{2n+1} = \dfrac{1}{(2n+1)\pi}$, $\sin\dfrac{1}{x_{2n+1}} = \sin(2n+1)\pi = 0$ and $\cos\dfrac{1}{x_{2n+1}} = \cos(2n+1)\pi = -1$, so $f'(x_{2n+1}) = x_{2n+1}^2 > 0$. Thus, f' changes sign infinitely often on both sides of 0.

Next, $g'(x_{2n}) = 8x_{2n}^3 + f'(x_{2n}) = 8x_{2n}^3 - x_{2n}^2 = x_{2n}^2(8x_{2n} - 1) < 0$ for $x_{2n} < \frac{1}{8}$, but $g'(x_{2n+1}) = 8x_{2n+1}^3 + x_{2n+1}^2 = x_{2n+1}^2(8x_{2n+1} + 1) > 0$ for $x_{2n+1} > -\frac{1}{8}$, so g' changes sign infinitely often on both sides of 0.

Last, $h'(x_{2n}) = -8x_{2n}^3 + f'(x_{2n}) = -8x_{2n}^3 - x_{2n}^2 = -x_{2n}^2(8x_{2n} + 1) < 0$ for $x_{2n} > -\frac{1}{8}$ and $h'(x_{2n+1}) = -8x_{2n+1}^3 + x_{2n+1}^2 = x_{2n+1}^2(-8x_{2n+1} + 1) > 0$ for $x_{2n+1} < \frac{1}{8}$, so h' changes sign infinitely often on both sides of 0.

(b) $f(0) = 0$ and since $\sin\frac{1}{x}$ and hence $x^4 \sin\frac{1}{x}$ is both positive and negative inifinitely often on both sides of 0, and arbitrarily close to 0, f has neither a local maximum nor a local minimum at 0.

Since $2 + \sin\frac{1}{x} \ge 1$, $g(x) = x^4\left(2 + \sin\frac{1}{x}\right) > 0$ for $x \ne 0$, so $g(0) = 0$ is a local minimum.

Since $-2 + \sin\frac{1}{x} \le -1$, $h(x) = x^4\left(-2 + \sin\frac{1}{x}\right) < 0$ for $x \ne 0$, so $h(0) = 0$ is a local maximum.