Section 3.5 Implicit Differentiation

44. If $x^2 + xy + y^3 = 1$, find the value of y''' at the point where $x = 1$.

Solution:

If $x = 1$ in $x^2 + xy + y^3 = 1$, then we get $1 + y + y^3 = 1 \quad \Rightarrow \quad y^3 + y = 0 \quad \Rightarrow \quad y(y^2+1) \quad \Rightarrow \quad y = 0$, so the point where $x = 1$ is $(1, 0)$. Differentiating implicitly with respect to x gives us $2x + xy' + y \cdot 1 + 3y^2 \cdot y' = 0$. Substituting 1 for x and 0 for y gives us $2 + y' + 0 + 0 = 0 \quad \Rightarrow \quad y' = -2$. Differentiating $2x + xy' + y + 3y^2y' = 0$ implicitly with respect to x gives us $2 + xy'' + y' \cdot 1 + y' + 3(y^2y'' + y' \cdot 2yy') = 0$. Now substitute 1 for x, 0 for y, and -2 for y'. $2 + y'' + (-2) + (-2) + 3(0+0) = 0 \quad \Rightarrow \quad y'' = 2$. Differentiating $2 + xy'' + 2y' + 3y^2y'' + 6y(y')^2 = 0$ implicitly with respect to x gives us $xy''' + y'' \cdot 1 + 2y'' + 3(y^2y''' + y'' \cdot 2yy') + 6[y \cdot 2y'y'' + (y')^2y'] = 0$. Now substitute 1 for x, 0 for y, -2 for y', and 2 for y''. $y''' + 2 + 4 + 3(0+0) + 6[0 + (-8)] = 0 \quad \Rightarrow \quad y''' = -2 - 4 + 48 = 42$.

48. Show by implicit differentiation that the tangent line to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point (x_0, y_0) has equation

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$$

Solution:

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \Rightarrow \quad \frac{2x}{a^2} + \frac{2yy'}{b^2} = 0 \quad \Rightarrow \quad y' = -\frac{b^2 x}{a^2 y} \quad \Rightarrow$ an equation of the tangent line at (x_0, y_0) is

$y - y_0 = \frac{-b^2 x_0}{a^2 y_0}(x - x_0)$. Multiplying both sides by $\frac{y_0}{b^2}$ gives $\frac{y_0 y}{b^2} - \frac{y_0^2}{b^2} = -\frac{x_0 x}{a^2} + \frac{x_0^2}{a^2}$. Since (x_0, y_0) lies on the ellipse, we have $\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1$.

65. Use implicit differentiation to find dy/dx for the equation

$$\frac{x}{y} = y^2 + 1 \qquad y \neq 0$$

and for the equivalent equation

$$x = y^3 + y \qquad y \neq 0$$

Show that although the expressions you get for dy/dx look different, they agree for all points that satisfy the given equation.

Solution:

For $\frac{x}{y} = y^2 + 1$, $y \neq 0$, we have $\frac{d}{dx}\left(\frac{x}{y}\right) = \frac{d}{dx}(y^2+1) \quad \Rightarrow \quad \frac{y \cdot 1 - x \cdot y'}{y^2} = 2y\,y' \quad \Rightarrow \quad y - x\,y' = 2y^3\,y' \quad \Rightarrow$ $2y^3\,y' + x\,y' = y \quad \Rightarrow \quad y'\,(2y^3 + x) = y \quad \Rightarrow \quad y' = \frac{y}{2y^3 + x}$.

For $x = y^3 + y$, $y \neq 0$, we have $\frac{d}{dx}(x) = \frac{d}{dx}(y^3 + y) \quad \Rightarrow \quad 1 = 3y^2y' + y' \quad \Rightarrow \quad 1 = y'\,(3y^2+1) \quad \Rightarrow$ $y' = \frac{1}{3y^2+1}$.

From part (a), $y' = \frac{y}{2y^3 + x}$. Since $y \neq 0$, we substitute $y^3 + y$ for x to get

$\frac{y}{2y^3 + x} = \frac{y}{2y^3 + (y^3 + y)} = \frac{y}{3y^3 + y} = \frac{y}{y(3y^2+1)} = \frac{1}{3y^2+1}$, which agrees with part (b).

66. The *Bessel function* of order 0, $y = J(x)$, satisfies the differential equation $xy'' + y' + xy = 0$ for all values of x and its value at 0 is $J(0) = 1$.

(a) Find $J'(0)$.

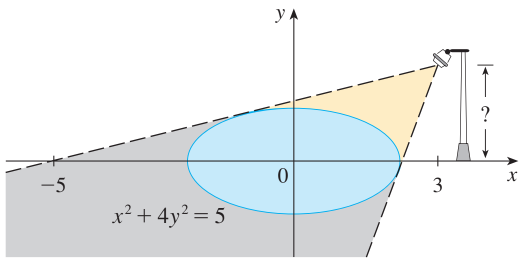
(b) Use implicit differentiation to find $J''(0)$.

Solution:

(a) $y = J(x)$ and $xy'' + y' + xy = 0 \quad \Rightarrow \quad xJ''(x) + J'(x) + xJ(x) = 0$. If $x = 0$, we have $0 + J'(0) + 0 = 0$, so $J'(0) = 0$.

(b) Differentiating $xy'' + y' + xy = 0$ implicitly, we get $xy''' + y'' \cdot 1 + y'' + xy' + y \cdot 1 = 0 \quad \Rightarrow$ $xy''' + 2y'' + xy' + y = 0$, so $xJ'''(x) + 2J''(x) + xJ'(x) + J(x) = 0$. If $x = 0$, we have $0 + 2J''(0) + 0 + 1 \quad [J(0) = 1 \text{ is given}] \quad = 0 \quad \Rightarrow \quad 2J''(0) = -1 \quad \Rightarrow \quad J''(0) = -\frac{1}{2}$.

67. The figure shows a lamp located three units to the right of the y-axis and a shadow created by the elliptical region $x^2 + 4y^2 \leq 5$. If the point $(-5, 0)$ is on the edge of the shadow, how far above the x-axis is the lamp located?



Solution:

$x^2 + 4y^2 = 5 \quad \Rightarrow \quad 2x + 4(2yy') = 0 \quad \Rightarrow \quad y' = -\dfrac{x}{4y}$. Now let h be the height of the lamp, and let (a, b) be the point of tangency of the line passing through the points $(3, h)$ and $(-5, 0)$. This line has slope $(h - 0)/[3 - (-5)] = \frac{1}{8}h$. But the slope of the tangent line through the point (a, b) can be expressed as $y' = -\dfrac{a}{4b}$, or as $\dfrac{b - 0}{a - (-5)} = \dfrac{b}{a + 5}$ [since the line passes through $(-5, 0)$ and (a, b)], so $-\dfrac{a}{4b} = \dfrac{b}{a + 5} \quad \Leftrightarrow \quad 4b^2 = -a^2 - 5a \quad \Leftrightarrow \quad a^2 + 4b^2 = -5a$. But $a^2 + 4b^2 = 5$ [since (a, b) is on the ellipse], so $5 = -5a \quad \Leftrightarrow \quad a = -1$. Then $4b^2 = -a^2 - 5a = -1 - 5(-1) = 4 \quad \Rightarrow \quad b = 1$, since the point is on the top half of the ellipse. So $\dfrac{h}{8} = \dfrac{b}{a + 5} = \dfrac{1}{-1 + 5} = \dfrac{1}{4} \quad \Rightarrow \quad h = 2$. So the lamp is located 2 units above the x-axis.