Section 2.5 Continuity

30. Explain, using Theorems 4, 5, 7, and 9, why the function is continuous at every number in its domain. State the domain.

$$B(u) = \sqrt{3u-2} + \sqrt[3]{2u-3}$$

Solution:

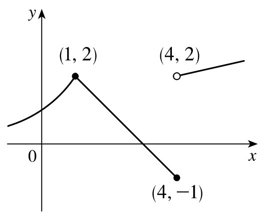
$B(u) = \sqrt{3u-2} + \sqrt[3]{2u-3}$ is defined when $3u - 2 \geq 0 \quad \Rightarrow \quad 3u \geq 2 \quad \Rightarrow \quad u \geq \frac{2}{3}$. (Note that $\sqrt[3]{2u-3}$ is defined everywhere.) So B has domain $\left[\frac{2}{3}, \infty\right)$. By Theorems 7 and 9, $\sqrt{3u-2}$ and $\sqrt[3]{2u-3}$ are each continuous on their domain because each is the composite of a root function and a polynomial function. B is the sum of these two functions, so it is continuous at every number in its domain by part 1 of Theorem 4.

44. Find the numbers at which f is discontinuous. At which of these numbers is f continuous from the right, from the left, or neither? Sketch the graph of f.

$$f(x) = \begin{cases} 2^x & \text{if } x \leq 1 \\ 3 - x & \text{if } 1 < x \leq 4 \\ \sqrt{x} & \text{if } x > 4 \end{cases}$$

Solution:

$$f(x) = \begin{cases} 2^x & \text{if } x \leq 1 \\ 3 - x & \text{if } 1 < x \leq 4 \\ \sqrt{x} & \text{if } x > 4 \end{cases}$$



f is continuous on $(-\infty, 1)$, $(1, 4)$, and $(4, \infty)$, where it is an exponential, a polynomial, and a root function, respectively.

Now $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 2^x = 2$ and $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (3-x) = 2$. Since $f(1) = 2$ we have continuity at 1. Also, $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} (3-x) = -1 = f(4)$ and $\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \sqrt{x} = 2$, so f is discontinuous at 4, but it is continuous from the left at 4.

48. Find the values of a and b that make f continuous everywhere.

$$f(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \leq x < 3 \\ 2x - a + b & \text{if } x \geq 3 \end{cases}$$

Solution:

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \leq x < 3 \\ 2x - a + b & \text{if } x \geq 3 \end{cases}$$

At $x = 2$: $\quad \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \dfrac{x^2-4}{x-2} = \lim_{x\to 2^-} \dfrac{(x+2)(x-2)}{x-2} = \lim_{x\to 2^-} (x+2) = 2 + 2 = 4$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (ax^2 - bx + 3) = 4a - 2b + 3$

We must have $4a - 2b + 3 = 4$, or $\mathbf{4a - 2b = 1}$ **(1)**.

At $x = 3$: $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (ax^2 - bx + 3) = 9a - 3b + 3$

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (2x - a + b) = 6 - a + b$

We must have $9a - 3b + 3 = 6 - a + b$, or $\mathbf{10a - 4b = 3}$ **(2)**.

Now solve the system of equations by adding -2 times equation **(1)** to equation **(2)**.

$$\begin{array}{rl} -8a + 4b = & -2 \\ 10a - 4b = & 3 \\ \hline 2a \qquad = & 1 \end{array}$$

So $a = \frac{1}{2}$. Substituting $\frac{1}{2}$ for a in **(1)** gives us $-2b = -1$, so $b = \frac{1}{2}$ as well. Thus, for f to be continuous on $(-\infty, \infty)$, $a = b = \frac{1}{2}$.

58. Use the Intermediate Value Theorem to show that there is a solution of the given equation in the specified interval.

$$\sin x = x^2 - x, \quad (1, 2)$$

Solution:

The equation $\sin x = x^2 - x$ is equivalent to the equation $\sin x - x^2 + x = 0$. $f(x) = \sin x - x^2 + x$ is continuous on the interval $[1, 2]$, $f(1) = \sin 1 \approx 0.84$, and $f(2) = \sin 2 - 2 \approx -1.09$. Since $\sin 1 > 0 > \sin 2 - 2$, there is a number c in $(1, 2)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\sin x - x^2 + x = 0$, or $\sin x = x^2 - x$, in the interval $(1, 2)$.

74. If a and b are positive numbers, prove that the equation

$$\frac{a}{x^3 + 2x^2 - 1} + \frac{b}{x^3 + x - 2} = 0$$

has at least one solution in the interval $(-1, 1)$.

Solution:

$\frac{a}{x^3 + 2x^2 - 1} + \frac{b}{x^3 + x - 2} = 0 \quad \Rightarrow \quad a(x^3 + x - 2) + b(x^3 + 2x^2 - 1) = 0$. Let $p(x)$ denote the left side of the last equation. Since p is continuous on $[-1, 1]$, $p(-1) = -4a < 0$, and $p(1) = 2b > 0$, there exists a c in $(-1, 1)$ such that $p(c) = 0$ by the Intermediate Value Theorem. Note that the only solution of either denominator that is in $(-1, 1)$ is $(-1 + \sqrt{5})/2 = r$, but $p(r) = (3\sqrt{5} - 9)a/2 \neq 0$. Thus, c is not a solution of either denominator, so $p(c) = 0 \quad \Rightarrow$ $x = c$ is a solution of the given equation.