Section 17.2 Nonhomogeneous Linear Equations

Review(p.1221-1222)

7. Solve the differential equation. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x\cos x$.

Solution:

$r^2 - 2r + 1 = 0 \quad \Rightarrow \quad r = 1$ and $y_c(x) = c_1e^x + c_2xe^x$. Try $y_p(x) = (Ax + B)\cos x + (Cx + D)\sin x \quad \Rightarrow$

$y_p' = (C - Ax - B)\sin x + (A + Cx + D)\cos x$ and $y_p'' = (2C - B - Ax)\cos x + (-2A - D - Cx)\sin x$. Substitution

gives $(-2Cx + 2C - 2A - 2D)\cos x + (2Ax - 2A + 2B - 2C)\sin x = x\cos x \quad \Rightarrow \quad A = 0$, $B = C = D = -\frac{1}{2}$.

The general solution is $y(x) = c_1e^x + c_2xe^x - \frac{1}{2}\cos x - \frac{1}{2}(x + 1)\sin x$.

8. Solve the differential equation. $\frac{d^2y}{dx^2} + 4y = \sin 2x$.

Solution:

$r^2 + 4 = 0 \quad \Rightarrow \quad r = \pm 2i$ and $y_c(x) = c_1\cos 2x + c_2\sin 2x$. Try $y_p(x) = Ax\cos 2x + Bx\sin 2x$ so that no term

of y_p is a solution of the complementary equation. Then $y_p' = (A + 2Bx)\cos 2x + (B - 2Ax)\sin 2x$ and

$y_p'' = (4B - 4Ax)\cos 2x + (-4A - 4Bx)\sin 2x$. Substitution gives $4B\cos 2x - 4A\sin 2x = \sin 2x \quad \Rightarrow$

$A = -\frac{1}{4}$ and $B = 0$. The general solution is $y(x) = c_1\cos 2x + c_2\sin 2x - \frac{1}{4}x\cos 2x$.

21. Assume that the earth is a solid sphere of uniform density with mass M and radius $R = 3960$ mi. For a particle of mass m within the earth at a distance r from the earth's center, the gravitational force attracting the particle to the center is

$$F_r = \frac{-GM_rm}{r^2}$$

where G is the gravitational constant and M_r is the mass of the earth within the sphere of radius r.

(a) Show that $F_r = \frac{-GMm}{R^3}r$.

(b) Suppose a hole is drilled through the earth along a diameter. Show that if a particle of mass m is dropped from rest at the surface, into the hole, then the distance $y = y(t)$ of the particle from the center of the earth at time t is given by

$$y''(t) = -k^2y(t)$$

where $k^2 = GM/R^3 = g/R$.

(c) Conclude from part (b) that the particle undergoes simple harmonic motion. Find the period T.

(d) With what speed does the particle pass through the center of the earth?

Solution:

(a) Since we are assuming that the earth is a solid sphere of uniform density, we can calculate the density ρ as follows:

$\rho = \dfrac{\text{mass of earth}}{\text{volume of earth}} = \dfrac{M}{\frac{4}{3}\pi R^3}$. If V_r is the volume of the portion of the earth which lies within a distance r of the center,

then $V_r = \frac{4}{3}\pi r^3$ and $M_r = \rho V_r = \dfrac{Mr^3}{R^3}$. Thus $F_r = -\dfrac{GM_r m}{r^2} = -\dfrac{GMm}{R^3}r$.

(b) The particle is acted upon by a varying gravitational force during its motion. By Newton's Second Law of Motion,

$m\,\dfrac{d^2y}{dt^2} = F_y = -\dfrac{GMm}{R^3}\,y$, so $y''(t) = -k^2 y\,(t)$ where $k^2 = \dfrac{GM}{R^3}$. At the surface, $-mg = F_R = -\dfrac{GMm}{R^2}$, so

$g = \dfrac{GM}{R^2}$. Therefore $k^2 = \dfrac{g}{R}$.

(c) The differential equation $y'' + k^2 y = 0$ has auxiliary equation $r^2 + k^2 = 0$. (This is the r of Section 17.1, not the r measuring distance from the earth's center.) The roots of the auxiliary equation are $\pm ik$, so the general solution of our differential equation for t is $y(t) = c_1 \cos kt + c_2 \sin kt$. It follows that $y'(t) = -c_1 k \sin kt + c_2 k \cos kt$. Now $y\,(0) = R$ and $y'(0) = 0$, so $c_1 = R$ and $c_2 k = 0$. Thus $y(t) = R\cos kt$ and $y'(t) = -kR\sin kt$. This is simple harmonic motion (see Section 17.3) with amplitude R, frequency k, and phase angle 0. The period is $T = 2\pi/k$.

$R \approx 6370\text{ km} = 6370 \times 10^6\text{ m}$ and $g = 9.8\text{ m/s}^2$, so $k = \sqrt{g/R} \approx 1.24 \times 10^{-3}\text{ s}^{-1}$ and

$T = 2\pi/k \approx 5079\text{ s} \approx 85\text{ min.}$

(d) $y(t) = 0 \quad \Leftrightarrow \quad \cos kt = 0 \quad \Leftrightarrow \quad kt = \frac{\pi}{2} + \pi n$ for some integer $n \quad \Rightarrow \quad y'(t) = -kR\sin\left(\frac{\pi}{2} + \pi n\right) = \pm kR$.

Thus the particle passes through the center of the earth with speed $kR \approx 7.899\text{ km/s} \approx 28{,}400\text{ km/h}$.