Section 17.1 Second-Order Linear Equations

4. Solve the differential equation. $y'' + y' - 12y = 0$.

Solution:

The auxiliary equation is $r^2 + r - 12 = 0 \quad \Rightarrow \quad (r-3)(r+4) = 0 \quad \Rightarrow \quad r = 3, r = -4$. Then by (8) the general solution is $y = c_1 e^{3x} + c_2 e^{-4x}$.

5. Solve the differential equation. $4y'' + 4y' + y = 0$.

Solution:

The auxiliary equation is $4r^2 + 4r + 1 = 0 \quad \Rightarrow \quad (2r+1)^2 = 0 \quad \Rightarrow \quad r = -\frac{1}{2}$. Then by (10), the general solution is $y = c_1 e^{-x/2} + c_2 x e^{-x/2}$.

10. Solve the differential equation. $3y'' + 4y' - 3y = 0$.

Solution:

The auxiliary equation is $3r^2 + 4r - 3 = 0 \quad \Rightarrow \quad r = \dfrac{-4 \pm \sqrt{52}}{6} = \dfrac{-2 \pm \sqrt{13}}{3}$, so

$y = c_1 e^{(-2+\sqrt{13})x/3} + c_2 e^{(-2-\sqrt{13})x/3}$.

34. If a, b, and c are all positive constants and $y(x)$ is a solution of the differential equation $ay'' + by' + cy = 0$, show that $\lim_{x\to\infty} y(x) = 0$.

Solution:

The auxiliary equation is $ar^2 + br + c = 0$. If $b^2 - 4ac > 0$, then any solution is of the form $y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$ where $r_1 = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $r_2 = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$. But a, b, and c are all positive so both r_1 and r_2 are negative and $\lim_{x\to\infty} y(x) = 0$. If $b^2 - 4ac = 0$, then any solution is of the form $y(x) = c_1 e^{rx} + c_2 x e^{rx}$ where $r = -b/(2a) < 0$ since a, b are positive. Hence $\lim_{x\to\infty} y(x) = 0$. Finally if $b^2 - 4ac < 0$, then any solution is of the form $y(x) = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$ where $\alpha = -b/(2a) < 0$ since a and b are positive. Thus $\lim_{x\to\infty} y(x) = 0$.