Section 14.8 Lagrange Multipliers

7. Each of these extreme value problems has a solution with both a maximum value and a minimum value. Use Lagrange multipliers to find the extreme values of the function subject to the given constraint.

$$f(x,y) = 2x^2 + 6y^2, \quad x^4 + 3y^4 = 1$$

Solution:

$f(x,y) = 2x^2 + 6y^2$, $g(x,y) = x^4 + 3y^4 = 1$, and $\nabla f = \lambda \nabla g \quad \Rightarrow \quad \langle 4x, 12y \rangle = \langle 4\lambda x^3, 12\lambda y^3 \rangle$, so we get the three equations $4x = 4\lambda x^3$, $12y = 12\lambda y^3$, and $x^4 + 3y^4 = 1$. The first equation implies that $x = 0$ or $x^2 = \dfrac{1}{\lambda}$. The second equation implies that $y = 0$ or $y^2 = \dfrac{1}{\lambda}$. Note that x and y cannot both be zero as this contradicts the third equation. If $x = 0$, the third equation implies $y = \pm\dfrac{1}{\sqrt[4]{3}}$. If $y = 0$, the third equation implies that $x = \pm 1$. Thus, f has possible extreme values at $\left(0, \pm\dfrac{1}{\sqrt[4]{3}}\right)$ and $(\pm 1, 0)$. Next, suppose $x^2 = y^2 = \dfrac{1}{\lambda}$. Then the third equation gives $\left(\dfrac{1}{\lambda}\right)^2 + 3\left(\dfrac{1}{\lambda}\right)^2 = 1 \quad \Rightarrow \quad \lambda = \pm 2$. $\lambda = -2$ results in a nonreal solution, so consider $\lambda = 2 \quad \Rightarrow \quad x = y = \pm\dfrac{1}{\sqrt{2}}$. Therefore, f also has possible extreme values at $\left(\pm\dfrac{1}{\sqrt{2}}, \pm\dfrac{1}{\sqrt{2}}\right)$ (all 4 combinations). Substituting all 8 points into f, we find the maximum value is $f\left(\pm\dfrac{1}{\sqrt{2}}, \pm\dfrac{1}{\sqrt{2}}\right) = 4$ and the minimum value is $f(\pm 1, 0) = 2$.

10. Each of these extreme value problems has a solution with both a maximum value and a minimum value. Use Lagrange multipliers to find the extreme values of the function subject to the given constraint.

$$f(x,y,z) = e^{xyz}; \quad 2x^2 + y^2 + z^2 = 24$$

Solution:

$f(x,y,z) = e^{xyz}$, $g(x,y,z) = 2x^2 + y^2 + z^2 = 24$, and $\nabla f = \lambda \nabla g \quad \Rightarrow \quad \langle yze^{xyz}, xze^{xyz}, xye^{xyz} \rangle = \langle 4\lambda x, 2\lambda y, 2\lambda z \rangle$. Then $yze^{xyz} = 4\lambda x$, $xze^{xyz} = 2\lambda y$, $xye^{xyz} = 2\lambda z$, and $2x^2 + y^2 + z^2 = 24$. If any of x, y, z, or λ is zero, then the first three equations imply that two of the variables x, y, z must be zero. If $x = y = z = 0$ it contradicts the fourth equation, so exactly two are zero, and from the fourth equation the possibilities are $(\pm 2\sqrt{3}, 0, 0)$, $(0, \pm 2\sqrt{6}, 0)$, $(0, 0, \pm 2\sqrt{6})$, all with an f-value of $e^0 = 1$. If none of x, y, z, λ is zero then from the first three equations we have $\dfrac{4\lambda x}{yz} = e^{xyz} = \dfrac{2\lambda y}{xz} = \dfrac{2\lambda z}{xy} \quad \Rightarrow \quad \dfrac{2x}{yz} = \dfrac{y}{xz} = \dfrac{z}{xy}$. This gives $2x^2 z = y^2 z \quad \Rightarrow \quad 2x^2 = y^2$ and $xy^2 = xz^2 \quad \Rightarrow$ $y^2 = z^2$. Substituting into the fourth equation, we have $y^2 + y^2 + y^2 = 24 \quad \Rightarrow \quad y^2 = 8 \quad \Rightarrow \quad y = \pm 2\sqrt{2}$, so $x^2 = 4 \quad \Rightarrow \quad x = \pm 2$ and $z^2 = y^2 \quad \Rightarrow \quad z = \pm 2\sqrt{2}$, giving possible points $(\pm 2, \pm 2\sqrt{2}, \pm 2\sqrt{2})$ (all combinations). The value of f is e^{16} when all coordinates are positive or exactly two are negative, and the value is e^{-16} when all are negative or exactly one of the coordinates is negative. Thus the maximum of f subject to the constraint is e^{16} and the minimum is e^{-16}.

28. Find the extreme values of f on the region described by the inequality.

$$f(x,y) = 2x^2 + 3y^2 - 4x - 5, \quad x^2 + y^2 \leq 16$$

Solution:

$f(x,y) = 2x^2 + 3y^2 - 4x - 5 \quad \Rightarrow \quad \nabla f = \langle 4x - 4, 6y \rangle = \langle 0, 0 \rangle \quad \Rightarrow \quad x = 1, y = 0$. Thus $(1,0)$ is the only critical point of f, and it lies in the region $x^2 + y^2 < 16$. On the boundary, $g(x,y) = x^2 + y^2 = 16 \quad \Rightarrow \quad \lambda \nabla g = \langle 2\lambda x, 2\lambda y \rangle$, so $6y = 2\lambda y \quad \Rightarrow$ either $y = 0$ or $\lambda = 3$. If $y = 0$, then $x = \pm 4$; if $\lambda = 3$, then $4x - 4 = 2\lambda x \quad \Rightarrow \quad x = -2$ and $y = \pm 2\sqrt{3}$. Now $f(1,0) = -7$, $f(4,0) = 11$, $f(-4,0) = 43$, and $f\left(-2, \pm 2\sqrt{3}\right) = 47$. Thus the maximum value of $f(x,y)$ on the disk $x^2 + y^2 \leq 16$ is $f\left(-2, \pm 2\sqrt{3}\right) = 47$, and the minimum value is $f(1,0) = -7$.

57. The plane $x + y + 2z = 2$ intersects the paraboloid $z = x^2 + y^2$ in an ellipse. Find the points on this ellipse that are nearest to and farthest from the origin.

Solution:

We need to find the extreme values of $f(x,y,z) = x^2 + y^2 + z^2$ subject to the two constraints $g(x,y,z) = x + y + 2z = 2$ and $h(x,y,z) = x^2 + y^2 - z = 0$. $\nabla f = \langle 2x, 2y, 2z \rangle$, $\lambda \nabla g = \langle \lambda, \lambda, 2\lambda \rangle$ and $\mu \nabla h = \langle 2\mu x, 2\mu y, -\mu \rangle$. Thus we need $2x = \lambda + 2\mu x$ **(1)**, $2y = \lambda + 2\mu y$ **(2)**, $2z = 2\lambda - \mu$ **(3)**, $x + y + 2z = 2$ **(4)**, and $x^2 + y^2 - z = 0$ **(5)**.

From **(1)** and **(2)**, $2(x - y) = 2\mu(x - y)$, so if $x \neq y$, $\mu = 1$. Putting this in **(3)** gives $2z = 2\lambda - 1$ or $\lambda = z + \frac{1}{2}$, but putting $\mu = 1$ into **(1)** says $\lambda = 0$. Hence $z + \frac{1}{2} = 0$ or $z = -\frac{1}{2}$. Then **(4)** and **(5)** become $x + y - 3 = 0$ and $x^2 + y^2 + \frac{1}{2} = 0$. The last equation cannot be true, so this case gives no solution. So we must have $x = y$. Then **(4)** and **(5)** become $2x + 2z = 2$ and $2x^2 - z = 0$ which imply $z = 1 - x$ and $z = 2x^2$. Thus $2x^2 = 1 - x$ or $2x^2 + x - 1 = (2x - 1)(x + 1) = 0$ so $x = \frac{1}{2}$ or $x = -1$. The two points to check are $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ and $(-1, -1, 2)$: $f\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = \frac{3}{4}$ and $f(-1,-1,2) = 6$. Thus $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ is the point on the ellipse nearest the origin and $(-1,-1,2)$ is the one farthest from the origin.

58. The plane $4x - 3y + 8z = 5$ intersects the cone $z^2 = x^2 + y^2$ in an ellipse.

(b) Use Lagrange multipliers to find the highest and lowest points on the ellipse.

Solution:

(b) We need to find the extreme values of $f(x,y,z) = z$ subject to the two constraints $g(x,y,z) = 4x - 3y + 8z = 5$ and $h(x,y,z) = x^2 + y^2 - z^2 = 0$.

$\nabla f = \lambda \nabla g + \mu \nabla h \quad \Rightarrow \quad \langle 0, 0, 1 \rangle = \lambda \langle 4, -3, 8 \rangle + \mu \langle 2x, 2y, -2z \rangle$, so we need $4\lambda + 2\mu x = 0 \quad \Rightarrow \quad x = -\frac{2\lambda}{\mu}$ **(1)**, $-3\lambda + 2\mu y = 0 \quad \Rightarrow \quad y = \frac{3\lambda}{2\mu}$ **(2)**, $8\lambda - 2\mu z = 1 \quad \Rightarrow \quad z = \frac{8\lambda - 1}{2\mu}$ **(3)**, $4x - 3y + 8z = 5$ **(4)**, and $x^2 + y^2 = z^2$ **(5)**. [Note that $\mu \neq 0$, else $\lambda = 0$ from **(1)**, but substitution into **(3)** gives a contradiction.]

Substituting **(1)**, **(2)**, and **(3)** into **(4)** gives $4\left(-\frac{2\lambda}{\mu}\right) - 3\left(\frac{3\lambda}{2\mu}\right) + 8\left(\frac{8\lambda - 1}{2\mu}\right) = 5 \quad \Rightarrow \quad \mu = \frac{39\lambda - 8}{10}$ and into **(5)** gives $\left(-\frac{2\lambda}{\mu}\right)^2 + \left(\frac{3\lambda}{2\mu}\right)^2 = \left(\frac{8\lambda - 1}{2\mu}\right)^2 \quad \Rightarrow \quad 16\lambda^2 + 9\lambda^2 = (8\lambda - 1)^2 \quad \Rightarrow \quad 39\lambda^2 - 16\lambda + 1 = 0 \quad \Rightarrow \quad \lambda = \frac{1}{13}$ or $\lambda = \frac{1}{3}$.

If $\lambda = \frac{1}{13}$ then $\mu = -\frac{1}{2}$ and $x = \frac{4}{13}$, $y = -\frac{3}{13}$, $z = \frac{5}{13}$. If $\lambda = \frac{1}{3}$ then $\mu = \frac{1}{2}$ and $x = -\frac{4}{3}$, $y = 1$, $z = \frac{5}{3}$. Thus, the highest point on the ellipse is $\left(-\frac{4}{3}, 1, \frac{5}{3}\right)$ and the lowest point is $\left(\frac{4}{13}, -\frac{3}{13}, \frac{5}{13}\right)$.