Section 14.6 Directional Derivatives and the Gradient Vector

12. Let $f(x,y,z) = y^2 e^{xyz}$, $P(0,1,-1)$, $\mathbf{u} = \left\langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13} \right\rangle$.

(a) Find the gradient of f.

(b) Evaluate the gradient at the point P.

(c) Find the rate of change of f at P in the direction of the vector $\mathbf{u}$.

Solution:

$f(x,y,z) = y^2 e^{xyz}$

(a) $\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \rangle = \left\langle y^2 e^{xyz}(yz), y^2 \cdot e^{xyz}(xz) + e^{xyz} \cdot 2y, y^2 e^{xyz}(xy) \right\rangle$

$= \left\langle y^3 z e^{xyz}, (xy^2 z + 2y)e^{xyz}, xy^3 e^{xyz} \right\rangle$

(b) $\nabla f(0,1,-1) = \langle -1, 2, 0 \rangle$

(c) $D_{\mathbf{u}} f(0,1,-1) = \nabla f(0,1,-1) \cdot \mathbf{u} = \langle -1, 2, 0 \rangle \cdot \left\langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13} \right\rangle = -\frac{3}{13} + \frac{8}{13} + 0 = \frac{5}{13}$

17. Find the directional derivative of the function at the given point in the direction of the vector $\mathbf{v}$.

$$f(x,y,z) = x^2 y + y^2 z, \quad (1,2,3), \quad \mathbf{v} = < 2, -1, 2 >$$

Solution:

$f(x,y,z) = x^2 y + y^2 z \quad \Rightarrow \quad \nabla f(x,y,z) = \left\langle 2xy, x^2 + 2yz, y^2 \right\rangle, \ \ \nabla f(1,2,3) = \langle 4, 13, 4 \rangle$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{4+1+4}} \langle 2, -1, 2 \rangle = \frac{1}{3} \langle 2, -1, 2 \rangle$, so

$D_{\mathbf{u}} f(1,2,3) = \nabla f(1,2,3) \cdot \mathbf{u} = \langle 4, 13, 4 \rangle \cdot \frac{1}{3} \langle 2, -1, 2 \rangle = \frac{1}{3}(8 - 13 + 8) = \frac{3}{3} = 1.$

32. Find the maximum rate of change of f at the given point and the direction in which it occurs.

$$f(p,q,r) = \arctan(pqr), \quad (1,2,1)$$

Solution:

$f(p,q,r) = \arctan(pqr) \quad \Rightarrow \quad \nabla f(p,q,r) = \left\langle \frac{qr}{1+(pqr)^2}, \frac{pr}{1+(pqr)^2}, \frac{pq}{1+(pqr)^2} \right\rangle, \nabla f(1,2,1) = \left\langle \frac{2}{5}, \frac{1}{5}, \frac{2}{5} \right\rangle$. Thus the maximum rate of change is $|\nabla f(1,2,1)| = \sqrt{\frac{4}{25} + \frac{1}{25} + \frac{4}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$ in the direction $\left\langle \frac{2}{5}, \frac{1}{5}, \frac{2}{5} \right\rangle$ or equivalently $\langle 2, 1, 2 \rangle$.

51. Find equations of (a) the tangent plane and (b) the normal line to the given surface at the specified point.

$$x + y + z = e^{xyz}, \quad (0,0,1)$$

Solution:

Let $F(x,y,z) = x + y + z - e^{xyz}$. Then $x + y + z = e^{xyz}$ is the level surface $F(x,y,z) = 0$, and $\nabla F(x,y,z) = \langle 1 - yze^{xyz}, 1 - xze^{xyz}, 1 - xye^{xyz} \rangle$.

(a) $\nabla F(0,0,1) = \langle 1, 1, 1 \rangle$ is a normal vector for the tangent plane at $(0,0,1)$, so an equation of the tangent plane is $1(x-0) + 1(y-0) + 1(z-1) = 0$ or $x + y + z = 1$.

(b) The normal line has direction $\langle 1, 1, 1 \rangle$, so parametric equations are $x = t, \ y = t, \ z = 1 + t$, and symmetric equations are $x = y = z - 1$.

57. Show that the equation of the tangent plane to the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ at the point (x_0, y_0, z_0) can be written as

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} + \frac{zz_0}{c^2} = 1$$

Solution:

$F(x, y, z) = x^2/a^2 + y^2/b^2 + z^2/c^2$. Then $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ is the level surface $F(x, y, z) = 1$ and

$\nabla F(x_0, y_0, z_0) = \left\langle \frac{2x_0}{a^2}, \frac{2y_0}{b^2}, \frac{2z_0}{c^2} \right\rangle$. Thus, an equation of the tangent plane at (x_0, y_0, z_0) is

$\frac{2x_0}{a^2}\,x + \frac{2y_0}{b^2}\,y + \frac{2z_0}{c^2}\,z = 2\left(\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2}\right) = 2(1) = 2$ since (x_0, y_0, z_0) is a point on the ellipsoid. Hence

$\frac{x_0}{a^2}\,x + \frac{y_0}{b^2}\,y + \frac{z_0}{c^2}\,z = 1$ is an equation of the tangent plane.

60. At what point on the ellipsoid $x^2 + y^2 + 2z^2 = 1$ is the tangent plane parallel to the plane $x + 2y + z = 1$? **Solution:**

Let $F(x, y, z) = x^2 + y^2 + 2z^2$; then the ellipsoid $x^2 + y^2 + 2z^2 = 1$ is a level surface of F. $\nabla F(x, y, z) = \langle 2x, 2y, 4z\rangle$ is a normal vector to the surface at (x, y, z) and so it is a normal vector for the tangent plane there. The tangent plane is parallel to the plane $x + 2y + z = 1$ when the normal vectors of the planes are parallel, so we need a point (x_0, y_0, z_0) on the ellipsoid where $\langle 2x_0, 2y_0, 4z_0\rangle = k\,\langle 1, 2, 1\rangle$ for some $k \neq 0$. Comparing components we have $2x_0 = k \;\Rightarrow\; x_0 = k/2$, $2y_0 = 2k \;\Rightarrow\; y_0 = k$, $4z_0 = k \;\Rightarrow\; z_0 = k/4$. $(x_0, y_0, z_0) = (k/2, k, k/4)$ lies on the ellipsoid, so $(k/2)^2 + k^2 + 2(k/4)^2 = 1 \;\Rightarrow\; \frac{11}{8}k^2 = 1 \;\Rightarrow\; k^2 = \frac{8}{11} \;\Rightarrow\; k = \pm 2\sqrt{\frac{2}{11}}$. Thus the tangent planes at the points $\left(\sqrt{\frac{2}{11}}, 2\sqrt{\frac{2}{11}}, \frac{1}{2}\sqrt{\frac{2}{11}}\right)$ and $\left(-\sqrt{\frac{2}{11}}, -2\sqrt{\frac{2}{11}}, -\frac{1}{2}\sqrt{\frac{2}{11}}\right)$ are parallel to the given plane.