Section 14.5 The Chain Rule

3. Use the Chain Rule to find dz/dt. $z = xy^3 - x^2y$, $x = t^2 + 1$, $y = t^2 - 1$.

Solution:

$z = xy^3 - x^2y, \quad x = t^2 + 1, \quad y = t^2 - 1 \quad \Rightarrow$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (y^3 - 2xy)(2t) + (3xy^2 - x^2)(2t) = 2t(y^3 - 2xy + 3xy^2 - x^2)$$

14. Use the Chain Rule to find $\partial z/\partial s$ and $\partial z/\partial t$.

$$z = \sqrt{x}e^{xy}, \quad x = 1 + st, \quad y = s^2 - t^2$$

Solution:

$z = \sqrt{x}\, e^{xy}, \quad x = 1 + st, \quad y = s^2 - t^2 \quad \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \left(\sqrt{x} \cdot e^{xy}(y) + e^{xy} \cdot \tfrac{1}{2}x^{-1/2}\right)(t) + \sqrt{x}\, e^{xy}(x)\,(2s) = \left(yt\sqrt{x} + \frac{t}{2\sqrt{x}} + 2x^{3/2}s\right)e^{xy}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \left(\sqrt{x} \cdot e^{xy}(y) + e^{xy} \cdot \tfrac{1}{2}x^{-1/2}\right)(s) + \sqrt{x}\, e^{xy}(x)\,(-2t) = \left(ys\sqrt{x} + \frac{s}{2\sqrt{x}} - 2x^{3/2}t\right)e^{xy}$$

38. Use Equations 6 to find $\partial z/\partial x$ and $\partial z/\partial y$. $yz + x\ln y = z^2$

Solution:

$yz + x\ln y = z^2$, so let $F(x, y, z) = yz + x\ln y - z^2 = 0$. Then $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{\ln y}{y - 2z} = \dfrac{\ln y}{2z - y}$ and

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{z + (x/y)}{y - 2z} = \frac{x + yz}{2yz - y^2}.$$

49. Assume that all the given functions are differentiable. If $z = f(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, (a) find $\partial z/\partial r$ and $\partial z/\partial\theta$ and (b) show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

Solution:

(a) By the Chain Rule, $\dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\cos\theta + \dfrac{\partial z}{\partial y}\sin\theta$, $\dfrac{\partial z}{\partial \theta} = \dfrac{\partial z}{\partial x}(-r\sin\theta) + \dfrac{\partial z}{\partial y}r\cos\theta$.

(b) $\left(\dfrac{\partial z}{\partial r}\right)^2 = \left(\dfrac{\partial z}{\partial x}\right)^2\cos^2\theta + 2\dfrac{\partial z}{\partial x}\dfrac{\partial z}{\partial y}\cos\theta\,\sin\theta + \left(\dfrac{\partial z}{\partial y}\right)^2\sin^2\theta$,

$\left(\dfrac{\partial z}{\partial \theta}\right)^2 = \left(\dfrac{\partial z}{\partial x}\right)^2 r^2\sin^2\theta - 2\dfrac{\partial z}{\partial x}\dfrac{\partial z}{\partial y}r^2\cos\theta\,\sin\theta + \left(\dfrac{\partial z}{\partial y}\right)^2 r^2\cos^2\theta$. Thus

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right](\cos^2\theta + \sin^2\theta) = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2.$$

52. If $u = f(x, y)$, where $x = e^s \cos t$ and $t = e^s \sin t$, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-2s}\left[\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2}\right]$$

Solution:

By the Chain Rule, $\dfrac{\partial u}{\partial s} = e^s \cos t\,\dfrac{\partial u}{\partial x} + e^s \sin t\,\dfrac{\partial u}{\partial y}$ and $\dfrac{\partial u}{\partial t} = -e^s \sin t\,\dfrac{\partial u}{\partial x} + e^s \cos t\,\dfrac{\partial u}{\partial y}$.

Then $\dfrac{\partial^2 u}{\partial s^2} = e^s \cos t\,\dfrac{\partial u}{\partial x} + e^s \cos t\,\dfrac{\partial}{\partial s}\left(\dfrac{\partial u}{\partial x}\right) + e^s \sin t\,\dfrac{\partial u}{\partial y} + e^s \sin t\,\dfrac{\partial}{\partial s}\left(\dfrac{\partial u}{\partial y}\right)$. But

$\dfrac{\partial}{\partial s}\left(\dfrac{\partial u}{\partial x}\right) = \dfrac{\partial^2 u}{\partial x^2}\dfrac{\partial x}{\partial s} + \dfrac{\partial^2 u}{\partial y\,\partial x}\dfrac{\partial y}{\partial s} = e^s \cos t\,\dfrac{\partial^2 u}{\partial x^2} + e^s \sin t\,\dfrac{\partial^2 u}{\partial y\,\partial x}$ and

$\dfrac{\partial}{\partial s}\left(\dfrac{\partial u}{\partial y}\right) = \dfrac{\partial^2 u}{\partial y^2}\dfrac{\partial y}{\partial s} + \dfrac{\partial^2 u}{\partial x\,\partial y}\dfrac{\partial x}{\partial s} = e^s \sin t\,\dfrac{\partial^2 u}{\partial y^2} + e^s \cos t\,\dfrac{\partial^2 u}{\partial x\,\partial y}$.

Also, by continuity of the partials, $\dfrac{\partial^2 u}{\partial x\,\partial y} = \dfrac{\partial^2 u}{\partial y\,\partial x}$. Thus,

$$\begin{aligned}
\frac{\partial^2 u}{\partial s^2} &= e^s \cos t\,\frac{\partial u}{\partial x} + e^s \cos t\left(e^s \cos t\,\frac{\partial^2 u}{\partial x^2} + e^s \sin t\,\frac{\partial^2 u}{\partial x\,\partial y}\right) + e^s \sin t\,\frac{\partial u}{\partial y} + e^s \sin t\left(e^s \sin t\,\frac{\partial^2 u}{\partial y^2} + e^s \cos t\,\frac{\partial^2 u}{\partial x\,\partial y}\right) \\
&= e^s \cos t\frac{\partial u}{\partial x} + e^s \sin t\,\frac{\partial u}{\partial y} + e^{2s}\cos^2 t\,\frac{\partial^2 u}{\partial x^2} + 2e^{2s}\cos t \sin t\,\frac{\partial^2 u}{\partial x\,\partial y} + e^{2s}\sin^2 t\,\frac{\partial^2 u}{\partial y^2}
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= -e^s \cos t\,\frac{\partial u}{\partial x} - e^s \sin t\,\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) - e^s \sin t\,\frac{\partial u}{\partial y} + e^s \cos t\,\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right) \\
&= -e^s \cos t\,\frac{\partial u}{\partial x} - e^s \sin t\left(-e^s \sin t\,\frac{\partial^2 u}{\partial x^2} + e^s \cos t\,\frac{\partial^2 u}{\partial x\,\partial y}\right) \\
&\qquad -e^s \sin t\,\frac{\partial u}{\partial y} + e^s \cos t\left(e^s \cos t\,\frac{\partial^2 u}{\partial y^2} - e^s \sin t\,\frac{\partial^2 u}{\partial x\,\partial y}\right) \\
&= -e^s \cos t\,\frac{\partial u}{\partial x} - e^s \sin t\,\frac{\partial u}{\partial y} + e^{2s}\sin^2 t\,\frac{\partial^2 u}{\partial x^2} - 2e^{2s}\cos t \sin t\,\frac{\partial^2 u}{\partial x\,\partial y} + e^{2s}\cos^2 t\,\frac{\partial^2 u}{\partial y^2}
\end{aligned}$$

Thus, $e^{-2s}\left(\dfrac{\partial^2 u}{\partial s^2} + \dfrac{\partial^2 u}{\partial t^2}\right) = (\cos^2 t + \sin^2 t)\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$, as desired.

59. Suppose that the equation $F(x, y, z) = 0$ implicitly defines each of the three variables x, y, and z as functions of the other two: $z = f(x, y)$, $y = g(x, z)$, $x = h(y, z)$. If F is differentiable and F_x, F_y, and F_z are all nonzero, show that

$$\frac{\partial z}{\partial x}\frac{\partial x}{\partial y}\frac{\partial y}{\partial z} = -1$$

Solution:

$F(x, y, z) = 0$ is assumed to define z as a function of x and y, that is, $z = f(x, y)$. So by (6), $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z}$ since $F_z \neq 0$.

Similarly, it is assumed that $F(x, y, z) = 0$ defines x as a function of y and z, that is $x = h(x, z)$. Then $F(h(y, z), y, z) = 0$ and by the Chain Rule, $F_x\dfrac{\partial x}{\partial y} + F_y\dfrac{\partial y}{\partial y} + F_z\dfrac{\partial z}{\partial y} = 0$. But $\dfrac{\partial z}{\partial y} = 0$ and $\dfrac{\partial y}{\partial y} = 1$, so $F_x\dfrac{\partial x}{\partial y} + F_y = 0 \;\Rightarrow\; \dfrac{\partial x}{\partial y} = -\dfrac{F_y}{F_x}$.

A similar calculation shows that $\dfrac{\partial y}{\partial z} = -\dfrac{F_z}{F_y}$. Thus, $\dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial y}\dfrac{\partial y}{\partial z} = \left(-\dfrac{F_x}{F_z}\right)\left(-\dfrac{F_y}{F_x}\right)\left(-\dfrac{F_z}{F_y}\right) = -1$.