Section 14.4 Tangent Planes and Linear Approximation

9. Find an equation of the tangent plane to the given surface at the specified point. $z = x\sin(x+y)$, $(-1,1,0)$

Solution:

$z = f(x,y) = x\sin(x+y) \quad \Rightarrow \quad f_x(x,y) = x\cdot\cos(x+y)\cdot 1 + \sin(x+y)\cdot 1 = x\cos(x+y)+\sin(x+y)$ and $f_y(x,y) = x\cos(x+y)\cdot 1$, so $f_x(-1,1) = (-1)\cos 0 + \sin 0 = -1$, $\;f_y(-1,1) = (-1)\cos 0 = -1$. Thus, an equation of the tangent plane is $z - 0 = f_x(-1,1)(x-(-1)) + f_y(-1,1)(y-1) \quad \Rightarrow \quad z = (-1)(x+1)+(-1)(y-1)$, or $x+y+z=0$.

20. Explain why the function is differentiable at the given point. Then find the linearization $L(x,y)$, yd of the function at that point.

$$f(x,y) = \frac{1+y}{1+x}, \quad (1,3)$$

Solution:

$f(x,y) = \dfrac{1+y}{1+x} = (1+y)(1+x)^{-1}$. The partial derivatives are $f_x(x,y) = (1+y)(-1)(1+x)^{-2} = -\dfrac{1+y}{(1+x)^2}$ and $f_y(x,y) = (1)(1+x)^{-1} = \dfrac{1}{1+x}$, so $f_x(1,3) = -1$ and $f_y(1,3) = \frac{1}{2}$. Both f_x and f_y are continuous functions for $x \neq -1$, so f is differentiable at $(1,3)$ by Theorem 8. The linearization of f at $(1,3)$ is

$L(x,y) = f(1,3) + f_x(1,3)(x-1) + f_y(1,3)(y-3) = 2 + (-1)(x-1) + \frac{1}{2}(y-3) = -x + \frac{1}{2}y + \frac{3}{2}$.

24. Verify the linear approximation at $(0,0)$. $\frac{y-1}{x+1} \approx x+y-1$

Solution:

Let $f(x,y) = \dfrac{y-1}{x+1}$. Then $f_x(x,y) = (y-1)(-1)(x+1)^{-2} = \dfrac{1-y}{(x+1)^2}$ and $f_y(x,y) = \dfrac{1}{x+1}$. Both f_x and f_y are continuous functions for $x \neq -1$, so by Theorem 8, f is differentiable at $(0,0)$. We have $f_x(0,0) = 1$, $\;f_y(0,0) = 1$ and the linear approximation of f at $(0,0)$ is $f(x,y) \approx f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) = -1 + 1x + 1y = x+y-1$.

52. Suppose you need to know an equation of the tangent plane to a surface S at the point $P(2,1,3)$. You don't have an equation for S but you know that the curves

$$\mathbf{r}_1(\mathrm{t}) = < 2+3t, 1-t^2, 3-4t+t^2 >$$
$$\mathbf{r}_2(\mathrm{u}) = < 1+u^2, 2u^3-1, 2u+1 >$$

both lie on S. Find an equation of the tangent plane at P.

Solution:

$\mathbf{r}_1(t) = \langle 2+3t, 1-t^2, 3-4t+t^2\rangle \quad \Rightarrow \quad \mathbf{r}_1'(t) = \langle 3, -2t, -4+2t\rangle$, $\quad \mathbf{r}_2(u) = \langle 1+u^2, 2u^3-1, 2u+1\rangle \quad \Rightarrow$ $\mathbf{r}_2'(u) = \langle 2u, 6u^2, 2\rangle$. Both curves pass through P since $\mathbf{r}_1(0) = \mathbf{r}_2(1) = \langle 2,1,3\rangle$, so the tangent vectors $\mathbf{r}_1'(0) = \langle 3,0,-4\rangle$ and $\mathbf{r}_2'(1) = \langle 2,6,2\rangle$ are both parallel to the tangent plane to S at P. A normal vector for the tangent plane is

$\mathbf{r}_1'(0) \times \mathbf{r}_2'(1) = \langle 3,0,-4\rangle \times \langle 2,6,2\rangle = \langle 24,-14,18\rangle$, so an equation of the tangent plane is

$24(x-2) - 14(y-1) + 18(z-3) = 0$ or $12x - 7y + 9z = 44$.

54. (a) The function

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

was graphed in Figure 4. Show that $f_x(0,0)$ and $f_y(0,0)$ both exist but f is not differentiable at $(0,0)$. [*Hint*: Use the result of Exercise 53.]

(b) Explain why f_x and f_y are not continuous at $(0,0)$.

Solution:

(a) $\lim_{h\to 0} \dfrac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \dfrac{0-0}{h} = 0$ and $\lim_{h\to 0} \dfrac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} \dfrac{0-0}{h} = 0$. Thus $f_x(0,0) = f_y(0,0) = 0$.

To show that f isn't differentiable at $(0,0)$ we need only show that f is not continuous at $(0,0)$ and apply Exercise 45. As $(x,y) \to (0,0)$ along the x-axis $f(x,y) = 0/x^2 = 0$ for $x \neq 0$ so $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along the x-axis. But as $(x,y) \to (0,0)$ along the line $y = x$, $f(x,x) = x^2/(2x^2) = \frac{1}{2}$ for $x \neq 0$ so $f(x,y) \to \frac{1}{2}$ as $(x,y) \to (0,0)$ along this line. Thus $\lim_{(x,y)\to(0,0)} f(x,y)$ doesn't exist, so f is discontinuous at $(0,0)$ and thus not differentiable there.

(b) For $(x,y) \neq (0,0)$, $f_x(x,y) = \dfrac{(x^2+y^2)y - xy(2x)}{(x^2+y^2)^2} = \dfrac{y(y^2-x^2)}{(x^2+y^2)^2}$. If we approach $(0,0)$ along the y-axis, then $f_x(x,y) = f_x(0,y) = \dfrac{y^3}{y^4} = \dfrac{1}{y}$, so $f_x(x,y) \to \pm\infty$ as $(x,y) \to (0,0)$. Thus $\lim_{(x,y)\to(0,0)} f_x(x,y)$ does not exist and $f_x(x,y)$ is not continuous at $(0,0)$. Similarly, $f_y(x,y) = \dfrac{(x^2+y^2)x - xy(2y)}{(x^2+y^2)^2} = \dfrac{x(x^2-y^2)}{(x^2+y^2)^2}$ for $(x,y) \neq (0,0)$, and if we approach $(0,0)$ along the x-axis, then $f_y(x,y) = f_x(x,0) = \dfrac{x^3}{x^4} = \dfrac{1}{x}$. Thus $\lim_{(x,y)\to(0,0)} f_y(x,y)$ does not exist and $f_y(x,y)$ is not continuous at $(0,0)$.