Section 14.3 Partial Derivatives

32. Find the first partial derivatives of the function. $u = x^{\frac{y}{z}}$.

Solution:

$$u = x^{y/z} \quad \Rightarrow \quad u_x = \frac{y}{z}\, x^{(y/z)-1}, \; u_y = x^{y/z} \ln x \cdot \frac{1}{z} = \frac{x^{y/z}}{z} \ln x, \;\; u_z = x^{y/z} \ln x \cdot \frac{-y}{z^2} = -\frac{y x^{y/z}}{z^2} \ln x$$

44. Use implicit differentiation to find $\partial z/\partial x$ and $\partial z/\partial y$.

$$yz + x \ln y = z^2$$

Solution:

$$yz + x \ln y = z^2 \quad \Rightarrow \quad \frac{\partial}{\partial x}(yz + x \ln y) = \frac{\partial}{\partial x}(z^2) \quad \Rightarrow \quad y\frac{\partial z}{\partial x} + \ln y = 2z\frac{\partial z}{\partial x} \quad \Rightarrow \quad \ln y = 2z\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial x} \quad \Rightarrow$$

$$\ln y = (2z - y)\frac{\partial z}{\partial x}, \text{ so } \frac{\partial z}{\partial x} = \frac{\ln y}{2z - y}.$$

$$\frac{\partial}{\partial y}(yz + x \ln y) = \frac{\partial}{\partial y}(z^2) \quad \Rightarrow \quad y\frac{\partial z}{\partial y} + z \cdot 1 + x \cdot \frac{1}{y} = 2z\frac{\partial z}{\partial y} \quad \Rightarrow \quad z + \frac{x}{y} = 2z\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial y} \quad \Rightarrow$$

$$z + \frac{x}{y} = (2z - y)\frac{\partial z}{\partial y}, \text{ so } \frac{\partial z}{\partial y} = \frac{z + (x/y)}{2z - y} = \frac{x + yz}{y(2z - y)}.$$

49. Find all the second partial derivatives.

$$z = \frac{y}{2x + 3y}$$

Solution:

$$z = \frac{y}{2x + 3y} = y(2x + 3y)^{-1} \quad \Rightarrow \quad z_x = y(-1)(2x + 3y)^{-2}(2) = -\frac{2y}{(2x + 3y)^2},$$

$$z_y = \frac{(2x + 3y) \cdot 1 - y \cdot 3}{(2x + 3y)^2} = \frac{2x}{(2x + 3y)^2}. \text{ Then } z_{xx} = -2y(-2)(2x + 3y)^{-3}(2) = \frac{8y}{(2x + 3y)^3},$$

$$z_{xy} = -\frac{(2x + 3y)^2 \cdot 2 - 2y \cdot 2(2x + 3y)(3)}{[(2x + 3y)^2]^2} = -\frac{(2x + 3y)\,(4x + 6y - 12y)}{(2x + 3y)^4} = \frac{6y - 4x}{(2x + 3y)^3},$$

$$z_{yx} = \frac{(2x + 3y)^2 \cdot 2 - 2x \cdot 2(2x + 3y)(2)}{[(2x + 3y)^2]^2} = \frac{6y - 4x}{(2x + 3y)^3}, \;\; z_{yy} = 2x(-2)(2x + 3y)^{-3}(3) = -\frac{12x}{(2x + 3y)^3}.$$

62. Find the indicated partial derivative(s). $V = \ln(r + s^2 + t^3)$; $\frac{\partial^3 V}{\partial r \partial s \partial t}$

Solution:

$$V = \ln(r + s^2 + t^3) \quad \Rightarrow \quad \frac{\partial V}{\partial t} = \frac{3t^2}{r + s^2 + t^3} = 3t^2(r + s^2 + t^3)^{-1},$$

$$\frac{\partial^2 V}{\partial s\, \partial t} = 3t^2(-1)(r + s^2 + t^3)^{-2}(2s) = -6st^2(r + s^2 + t^3)^{-2},$$

$$\frac{\partial^3 V}{\partial r\, \partial s\, \partial t} = -6st^2(-2)(r + s^2 + t^3)^{-3}(1) = 12st^2(r + s^2 + t^3)^{-3} = \frac{12st^2}{(r + s^2 + t^3)^3}.$$

67. If $f(x,y,z) = xy^2z^3 + \arcsin(x\sqrt{z})$, find f_{xzy}. [*Hint*:Which order of differentiation is easiest?]

Solution:

Assuming that the third partial derivatives of f are continuous (easily verified), we can write $f_{xzy} = f_{yxz}$. Then

$f(x,y,z) = xy^2z^3 + \arcsin\left(x\sqrt{z}\right) \quad \Rightarrow \quad f_y = 2xyz^3 + 0$, $f_{yx} = 2yz^3$, and $f_{yxz} = 6yz^2 = f_{xzy}$.