Section 14.2 Limits and Continuity

17. Show that the limit does not exist. $\lim\limits_{(x,y)\to(0,0)} \frac{y^2 \sin^2 x}{x^4+y^4}$.

Solution:

$f(x,y) = \dfrac{y^2 \sin^2 x}{x^4+y^4}$. First approach $(0,0)$ along the y-axis. Then $f(0,y) = 0/y^4 = 0$ for $y \neq 0$, so $f(x,y) \to 0$. Now approach $(0,0)$ along the line $y = x$. Then $f(x,x) = \dfrac{x^2 \sin^2 x}{2x^4} = \dfrac{1}{2}\left(\dfrac{\sin x}{x}\right)^2$ for $x \neq 0$, so by Equation 3.3.5, $f(x,y) \to \frac{1}{2}(1)^2 = \frac{1}{2}$. Since f has two different limits along two different lines, the limit does not exist.

23. Find the limit, if it exists, or show that the limit does not exist. $\lim\limits_{(x,y)\to(0,0)} \frac{xy^2 \cos y}{x^2+y^4}$.

Solution:

Let $f(x,y) = \dfrac{xy^2 \cos y}{x^2+y^4}$. Then $f(x,0) = 0$ for $x \neq 0$, so $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along the x-axis. Approaching $(0,0)$ along the y-axis or the line $y = x$ also gives a limit of 0. But $f\left(y^2, y\right) = \dfrac{y^2 y^2 \cos y}{(y^2)^2 + y^4} = \dfrac{y^4 \cos y}{2y^4} = \dfrac{\cos y}{2}$ for $y \neq 0$, so $f(x,y) \to \frac{1}{2}\cos 0 = \frac{1}{2}$ as $(x,y) \to (0,0)$ along the parabola $x = y^2$. Thus the limit doesn't exist.

33. Use the Squeeze Theorem to find the limit. $\lim\limits_{(x,y)\to(0,0)} \frac{xy^4}{x^4+y^4}$.

Solution:

We use the Squeeze Theorem to show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^4}{x^4+y^4} = 0$:

$0 \leq \dfrac{|x|\, y^4}{x^4+y^4} \leq |x|$ since $0 \leq \dfrac{y^4}{x^4+y^4} \leq 1$, and $|x| \to 0$ as $(x,y) \to (0,0)$, so $\dfrac{|x|\, y^4}{x^4+y^4} \to 0 \quad \Rightarrow \quad \dfrac{xy^4}{x^4+y^4} \to 0$ as $(x,y) \to (0,0)$.

52. Use polar coordinates to find the limit. [If (r,θ) are polar coordinates of the point (x,y) with $r \geq 0$, note that $r \to 0^+$ as $(x,y) \to (0,0)$.]

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$$

Solution:

$$\begin{aligned}\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2) &= \lim_{r\to 0^+} r^2 \ln r^2 = \lim_{r\to 0^+} \frac{\ln r^2}{1/r^2} = \lim_{r\to 0^+} \frac{(1/r^2)(2r)}{-2/r^3} \quad \text{[using l'Hospital's Rule]} \\ &= \lim_{r\to 0^+} (-r^2) = 0\end{aligned}$$