# Section 9.5 Linear Equations

24. Solve the initial-value problem.

$$(x^2+1)\frac{dy}{dx}+3x(y-1)=0, \quad y(0)=2$$

**Solution:**

$(x^2+1)\,\dfrac{dy}{dx}+3x(y-1)=0 \quad\Rightarrow\quad (x^2+1)\,y'+3xy=3x \quad\Rightarrow\quad y'+\dfrac{3x}{x^2+1}\,y=\dfrac{3x}{x^2+1}.$

$I(x)=e^{\int 3x/(x^2+1)\,dx}=e^{(3/2)\ln|x^2+1|}=\left(e^{\ln(x^2+1)}\right)^{3/2}=(x^2+1)^{3/2}$. Multiplying by $(x^2+1)^{3/2}$ gives

$(x^2+1)^{3/2}\,y'+3x(x^2+1)^{1/2}\,y=3x(x^2+1)^{1/2} \quad\Rightarrow\quad \left[(x^2+1)^{3/2}y\right]'=3x(x^2+1)^{1/2} \quad\Rightarrow$

$(x^2+1)^{3/2}\,y=\int 3x(x^2+1)^{1/2}\,dx=(x^2+1)^{3/2}+C \quad\Rightarrow\quad y=1+C(x^2+1)^{-3/2}$. Since $y(0)=2$, we have

$2=1+C(1) \quad\Rightarrow\quad C=1$ and hence, $y=1+(x^2+1)^{-3/2}$.

30. Solve the second-order equation $xy''+2y'=12x^2$ by making the substitution $u=y'$.

**Solution:**

$xy''+2y'=12x^2$ and $u=y' \quad\Rightarrow\quad xu'+2u=12x^2 \quad\Rightarrow\quad u'+\dfrac{2}{x}\,u=12x.$

$I(x)=e^{\int(2/x)\,dx}=e^{2\ln|x|}=\left(e^{\ln|x|}\right)^2=|x|^2=x^2$. Multiplying the last differential equation by $x^2$ gives

$x^2u'+2xu=12x^3 \quad\Rightarrow\quad (x^2u)'=12x^3 \quad\Rightarrow\quad x^2u=\int 12x^3\,dx=3x^4+C \quad\Rightarrow\quad u=3x^2+C/x^2 \quad\Rightarrow$

$y'=3x^2+C/x^2 \quad\Rightarrow\quad y=x^3-C/x+D.$

39. An object with mass $m$ is dropped from rest and we assume that the air resistance is proportional to the speed of the object. If $s(t)$ is the distance dropped after $t$ seconds, then the speed is $v=s'(t)$ and the acceleration is $a=v'(t)$. If $g$ is the acceleration due to gravity, then the downward force on the object is $mg-cv$, where $c$ is a positive constant, and Newton's Second Law gives

$$m\frac{dv}{dt}=mg-cv$$

(a) Solve this as a linear equation to show that

$$v=\frac{mg}{c}(1-e^{-ct/m})$$

(b) What is the limiting velocity?

(c) Find the distance the object has fallen after t seconds.

**Solution:**

(a) $m\,\dfrac{dv}{dt}=mg-cv \quad\Rightarrow\quad \dfrac{dv}{dt}+\dfrac{c}{m}v=g$ and $I(t)=e^{\int(c/m)\,dt}=e^{(c/m)t}$, and multiplying the last differential

equation by $I(t)$ gives $e^{(c/m)t}\,\dfrac{dv}{dt}+\dfrac{vce^{(c/m)t}}{m}=ge^{(c/m)t} \quad\Rightarrow\quad \left[e^{(c/m)t}v\right]'=ge^{(c/m)t}$. Hence,

$v(t)=e^{-(c/m)t}\left[\int ge^{(c/m)t}\,dt+K\right]=mg/c+Ke^{-(c/m)t}$. But the object is dropped from rest, so $v(0)=0$ and

$K=-mg/c$. Thus, the velocity at time $t$ is $v(t)=(mg/c)[1-e^{-(c/m)t}]$.

(b) $\lim\limits_{t\to\infty} v(t)=mg/c$

(c) $s(t)=\int v(t)\,dt=(mg/c)[t+(m/c)e^{-(c/m)t}]+c_1$ where $c_1=s(0)-m^2g/c^2$.

$s(0)$ is the initial position, so $s(0)=0$ and $s(t)=(mg/c)[t+(m/c)e^{-(c/m)t}]-m^2g/c^2$.

41. (a) Show that the substitution $z = 1/P$ transforms the logistic differential equation $P' = kP(1 - P/M)$ into the linear differential equation

$$z' + kz = \frac{k}{M}$$

(b) Solve the linear differential equation in part (a) and thus obtain an expression for $P(t)$. Compare with Equation 9.4.7.

**Solution:**

(a) $z = \frac{1}{P} \quad \Rightarrow \quad P = \frac{1}{z} \quad \Rightarrow \quad P' = -\frac{z'}{z^2}$. Substituting into $P' = kP(1 - P/M)$ gives us $-\frac{z'}{z^2} = k\,\frac{1}{z}\left(1 - \frac{1}{zM}\right) \quad \Rightarrow$

$z' = -kz\left(1 - \frac{1}{zM}\right) \quad \Rightarrow \quad z' = -kz + \frac{k}{M} \quad \Rightarrow \quad z' + kz = \frac{k}{M}$ $(\star)$.

(b) The integrating factor is $e^{\int k\,dt} = e^{kt}$. Multiplying $(\star)$ by $e^{kt}$ gives $e^{kt}z' + ke^{kt}z = \frac{ke^{kt}}{M} \quad \Rightarrow \quad (e^{kt}z)' = \frac{k}{M}\,e^{kt} \quad \Rightarrow$

$e^{kt}z = \int \frac{k}{M}\,e^{kt}\,dt \quad \Rightarrow \quad e^{kt}z = \frac{1}{M}\,e^{kt} + C \quad \Rightarrow \quad z = \frac{1}{M} + Ce^{-kt}$. Since $P = \frac{1}{z}$, we have

$P = \dfrac{1}{\frac{1}{M} + Ce^{-kt}} \quad \Rightarrow \quad P = \dfrac{M}{1 + MCe^{-kt}}$, which agrees with Equation 9.4.7, $P = \dfrac{M}{1 + Ae^{-kt}}$, when $MC = A$.