Section 9.4 Models for Population Growth

9. The population of the world was about 6.1 billion in 2000. Birth rates around that time ranged from 35 to 40 million per year and death rates ranged from 15 to 20 million per year. Let's assume that the carrying capacity for world population is 20 billion.

(a) Write the logistic differential equation for these data. (Because the initial population is small compared to the carrying capacity, you can take k to be an estimate of the initial relative growth rate.)

(b) Use the logistic model to estimate the world population in the year 2010 and compare with the actual population of 6.9 billion.

(c) Use the logistic model to predict the world population in the years 2100 and 2500.

Solution:

(a) We will assume that the difference in birth and death rates is 20 million/year. Let $t = 0$ correspond to the year 2000. Thus,

$$k \approx \frac{1}{P}\frac{dP}{dt} = \frac{1}{6.1 \text{ billion}}\left(\frac{20 \text{ million}}{\text{year}}\right) = \frac{1}{305}, \text{ and } \frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) = \frac{1}{305}P\left(1 - \frac{P}{20}\right) \text{ with } P \text{ in billions.}$$

(b) $A = \frac{M - P_0}{P_0} = \frac{20 - 6.1}{6.1} = \frac{139}{61} \approx 2.2787.$ $P(t) = \frac{M}{1 + Ae^{-kt}} = \frac{20}{1 + \frac{139}{61}e^{-t/305}}$, so

$P(10) = \frac{20}{1 + \frac{139}{61}e^{-10/305}} \approx 6.24$ billion, which underestimates the actual 2010 population of 6.9 billion.

(c) The years 2100 and 2500 correspond to $t = 100$ and $t = 500$, respectively. $P(100) = \frac{20}{1 + \frac{139}{61}e^{-100/305}} \approx 7.57$ billion

and $P(500) = \frac{20}{1 + \frac{139}{61}e^{-500/305}} \approx 13.87$ billion.

18. **Doomsday Equation** Let c be a positive number. A differential equation of the form

$$\frac{dy}{dt} = ky^{1+c}$$

where k is a positive constant, is called a *doomsday equation* because the exponent in the expression ky^{1+c} is larger than the exponent 1 for natural growth.

(a) Determine the solution that satisfies the initial condition $y(0) = y_0$.

(b) Show that there is a finite time $t = T$ (doomsday) such that $\lim_{t\to T^-} y(t) = \infty$.

(c) An especially prolific breed of rabbits has the growth term $ky^{1.01}$. If 2 such rabbits breed initially and the warren has 16 rabbits after three months, then when is doomsday?

Solution:

(a) $\frac{dy}{dt} = ky^{1+c} \Rightarrow y^{-1-c}\,dy = k\,dt \Rightarrow \frac{y^{-c}}{-c} = kt + C$. Since $y(0) = y_0$, we have $C = \frac{y_0^{-c}}{-c}$. Thus,

$\frac{y^{-c}}{-c} = kt + \frac{y_0^{-c}}{-c}$, or $y^{-c} = y_0^{-c} - ckt$. So $y^c = \frac{1}{y_0^{-c} - ckt} = \frac{y_0^c}{1 - cy_0^c kt}$ and $y(t) = \frac{y_0}{(1 - cy_0^c kt)^{1/c}}$.

(b) $y(t) \to \infty$ as $1 - cy_0^c kt \to 0$, that is, as $t \to \frac{1}{cy_0^c k}$. Define $T = \frac{1}{cy_0^c k}$. Then $\lim_{t\to T^-} y(t) = \infty$.

(c) According to the data given, we have $c = 0.01$, $y(0) = 2$, and $y(3) = 16$, where the time t is given in months. Thus,

$y_0 = 2$ and $16 = y(3) = \frac{y_0}{(1 - cy_0^c k \cdot 3)^{1/c}}$. Since $T = \frac{1}{cy_0^c k}$, we will solve for $cy_0^c k$. $16 = \frac{2}{(1 - 3cy_0^c k)^{100}} \Rightarrow$

$1 - 3cy_0^c k = \left(\frac{1}{8}\right)^{0.01} = 8^{-0.01} \Rightarrow cy_0^c k = \frac{1}{3}\left(1 - 8^{-0.01}\right)$. Thus, doomsday occurs when

$t = T = \frac{1}{cy_0^c k} = \frac{3}{1 - 8^{-0.01}} \approx 145.77$ months or 12.15 years.

21. There is considerable evidence to support the theory that for some species there is a minimum population m such that the species will become extinct if the size of the population falls below m. This condition can be incorporated into the logistic equation by introducing the factor $(1 - m/P)$. Thus the modified logistic model is given by the differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)\left(1 - \frac{m}{P}\right)$$

(a) Use the differential equation to show that any solution is increasing if $m < P < M$ and decreasing if $0 < P < m$.

(b) For the case where $k = 0.08$, $M = 1000$, and $m = 200$, draw a direction field and use it to sketch several solution curves. Describe what happens to the population for various initial populations. What are the equilibrium solutions?

(c) Solve the differential equation explicitly, either by using partial fractions or with a computer algebra system. Use the initial population P_0.

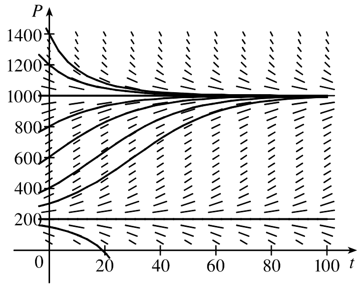
(d) Use the solution in part (c) to show that if $P_0 < m$, then the species will become extinct. [*Hint*: Show that the numerator in your expression for $P(t)$ is 0 for some value of t.]

Solution:

(a) $\frac{dP}{dt} = (kP)\left(1 - \frac{P}{M}\right)\left(1 - \frac{m}{P}\right)$. If $m < P < M$, then $dP/dt = (+)(+)(+) = + \Rightarrow P$ is increasing.

If $0 < P < m$, then $dP/dt = (+)(+)(-) = - \Rightarrow P$ is decreasing.

(b)



$k = 0.08$, $M = 1000$, and $m = 200 \Rightarrow$

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right)\left(1 - \frac{200}{P}\right)$$

For $0 < P_0 < 200$, the population dies out. For $P_0 = 200$, the population is steady. For $200 < P_0 < 1000$, the population increases and approaches 1000. For $P_0 > 1000$, the population decreases and approaches 1000.

The equilibrium solutions are $P(t) = 200$ and $P(t) = 1000$.

(c) $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)\left(1 - \frac{m}{P}\right) = kP\left(\frac{M-P}{M}\right)\left(\frac{P-m}{P}\right) = \frac{k}{M}(M-P)(P-m) \Leftrightarrow$

$\int \frac{dP}{(M-P)(P-m)} = \int \frac{k}{M}\,dt$. By partial fractions, $\frac{1}{(M-P)(P-m)} = \frac{A}{M-P} + \frac{B}{P-m}$, so $A(P-m) + B(M-P) = 1$.

If $P = m$, $B = \frac{1}{M-m}$; if $P = M$, $A = \frac{1}{M-m}$, so $\frac{1}{M-m}\int\left(\frac{1}{M-P} + \frac{1}{P-m}\right)dP = \int \frac{k}{M}\,dt \Rightarrow$

$\frac{1}{M-m}\left(-\ln|M-P| + \ln|P-m|\right) = \frac{k}{M}t + C \Rightarrow \frac{1}{M-m}\ln\left|\frac{P-m}{M-P}\right| = \frac{k}{M}t + C \Rightarrow$

$\ln\left|\frac{P-m}{M-P}\right| = (M-m)\frac{k}{M}t + C_1 \Leftrightarrow \frac{P-m}{M-P} = De^{(M-m)(k/M)t}$ $[D = \pm e^{C_1}]$.

Let $t = 0$: $\frac{P_0 - m}{M - P_0} = D$. So $\frac{P-m}{M-P} = \frac{P_0-m}{M-P_0}e^{(M-m)(k/M)t}$.

Solving for P, we get $P(t) = \frac{m(M-P_0) + M(P_0-m)e^{(M-m)(k/M)t}}{M - P_0 + (P_0-m)e^{(M-m)(k/M)t}}$.

(d) If $P_0 < m$, then $P_0 - m < 0$. Let $N(t)$ be the numerator of the expression for $P(t)$ in part (c). Then $N(0) = P_0(M-m) > 0$, and $P_0 - m < 0 \Leftrightarrow \lim_{t\to\infty} M(P_0-m)e^{(M-m)(k/M)t} = -\infty \Rightarrow \lim_{t\to\infty} N(t) = -\infty$.

Since N is continuous, there is a number t such that $N(t) = 0$ and thus $P(t) = 0$. So the species will become extinct.