Section 9.3 Separable Equations

16. Find the solution of the differential equation that satisfies the given initial condition.

$$x^2 y' = k \sec y, \quad y(1) = \pi/6$$

Solution:

$x^2\, y' = k \sec y \quad \Rightarrow \quad x^2 \dfrac{dy}{dx} = \dfrac{k}{\cos y} \quad \Rightarrow \quad \cos y\, dy = k\, \dfrac{dx}{x^2} \quad \Rightarrow \quad \int \cos y\, dy = \int kx^{-2} dx \quad \Rightarrow \quad \sin y = -kx^{-1} + C.$

$y(1) = \frac{\pi}{6} \quad \Rightarrow \quad \sin \frac{\pi}{6} = -k(1)^{-1} + C \quad \Rightarrow \quad \frac{1}{2} = -k + C \quad \Rightarrow \quad C = \frac{1}{2} + k$, so

$\sin y = -\dfrac{k}{x} + \dfrac{1}{2} + k \quad \Rightarrow \quad y = \sin^{-1}\left(-\dfrac{k}{x} + \dfrac{1}{2} + k\right).$

18. Find the solution of the differential equation that satisfies the given initial condition.

$$x + 3y^2 \sqrt{x^2+1} \frac{dy}{dx} = 0, \quad y(0) = 1$$

Solution:

$x + 3y^2\sqrt{x^2+1}\, \dfrac{dy}{dx} = 0 \quad \Rightarrow \quad 3y^2\sqrt{x^2+1}\, \dfrac{dy}{dx} = -x \quad \Rightarrow \quad 3y^2\, dy = \dfrac{-x}{\sqrt{x^2+1}}\, dx \quad \Rightarrow$

$\int 3y^2\, dy = \int -x(x^2+1)^{-1/2}\, dx \quad \Rightarrow \quad y^3 = -(x^2+1)^{1/2} + C. \quad y(0) = 1 \quad \Rightarrow \quad 1^3 = -(0^2+1)^{1/2} + C \quad \Rightarrow$

$C = 2$, so $y^3 = -(x^2+1)^{1/2} + 2 \quad \Rightarrow \quad y = (2 - \sqrt{x^2+1})^{1/3}.$

42. In an elementary chemical reaction, single molecules of two reactants A and B form a molecule of the product C: A + B $\rightarrow$ C. The law of mass action states that the rate of reaction is proportional to the product of the concentrations of A and B:

$$\frac{d[\mathrm{C}]}{dt} = k[\mathrm{A}][\mathrm{B}]$$

(See Example 3.7.4.) Thus, if the initial concentrations are [A]= a moles/L and [B]= b moles/L and we write $x =$[C], then we have

$$\frac{dx}{dt} = k(a-x)(b-x)$$

(a) Assuming that $a \neq b$, find x as a function of t. Use the fact that the initial concentration of C is 0.

(b) Find $x(t)$ assuming that $a = b$. How does this expression for $x(t)$ simplify if it is known that [C]= $\frac{1}{2}a$ after 20 seconds?

Solution:

(a) $\dfrac{dx}{dt} = k(a-x)(b-x),\ a \neq b$. Using partial fractions, $\dfrac{1}{(a-x)(b-x)} = \dfrac{1/(b-a)}{a-x} - \dfrac{1/(b-a)}{b-x}$, so

$$\int \frac{dx}{(a-x)(b-x)} = \int k\,dt \quad\Rightarrow\quad \frac{1}{b-a}\left(-\ln|a-x| + \ln|b-x|\right) = kt + C \quad\Rightarrow\quad \ln\left|\frac{b-x}{a-x}\right| = (b-a)(kt+C).$$

The concentrations [A] $= a - x$ and [B] $= b - x$ cannot be negative, so $\dfrac{b-x}{a-x} \geq 0$ and $\left|\dfrac{b-x}{a-x}\right| = \dfrac{b-x}{a-x}$.

We now have $\ln\left(\dfrac{b-x}{a-x}\right) = (b-a)(kt+C)$. Since $x(0) = 0$, we get $\ln\left(\dfrac{b}{a}\right) = (b-a)C$. Hence,

$$\ln\left(\frac{b-x}{a-x}\right) = (b-a)kt + \ln\left(\frac{b}{a}\right) \quad\Rightarrow\quad \frac{b-x}{a-x} = \frac{b}{a}e^{(b-a)kt} \quad\Rightarrow\quad x = \frac{b[e^{(b-a)kt}-1]}{be^{(b-a)kt}/a - 1} = \frac{ab[e^{(b-a)kt}-1]}{be^{(b-a)kt}-a}\ \frac{\text{moles}}{\text{L}}.$$

(b) If $b = a$, then $\dfrac{dx}{dt} = k(a-x)^2$, so $\displaystyle\int \frac{dx}{(a-x)^2} = \int k\,dt$ and $\dfrac{1}{a-x} = kt + C$. Since $x(0) = 0$, we get $C = \dfrac{1}{a}$.

Thus, $a - x = \dfrac{1}{kt + 1/a}$ and $x = a - \dfrac{a}{akt+1} = \dfrac{a^2kt}{akt+1}\ \dfrac{\text{moles}}{\text{L}}$. Suppose $x = [\text{C}] = a/2$ when $t = 20$. Then

$$x(20) = a/2 \quad\Rightarrow\quad \frac{a}{2} = \frac{20a^2k}{20ak+1} \quad\Rightarrow\quad 40a^2k = 20a^2k + a \quad\Rightarrow\quad 20a^2k = a \quad\Rightarrow\quad k = \frac{1}{20a}, \text{ so}$$

$$x = \frac{a^2t/(20a)}{1 + at/(20a)} = \frac{at/20}{1+t/20} = \frac{at}{t+20}\ \frac{\text{moles}}{\text{L}}.$$

44. A sphere with radius 1 m has temperature 15 °C. It lies inside a concentric sphere with radius 2 m and temperature 25 °C. The temperature $T(r)$ at a distance r from the common center of the spheres satisfies the differential equation

$$\frac{d^2T}{dr^2} + \frac{2}{r}\frac{dT}{dr} = 0$$

If we let $S = dT/dr$, then S satisfies a first-order differential equation. Solve it to find an expression for the temperature $T(r)$ between the spheres.

Solution:

If $S = \dfrac{dT}{dr}$, then $\dfrac{dS}{dr} = \dfrac{d^2T}{dr^2}$. The differential equation $\dfrac{d^2T}{dr^2} + \dfrac{2}{r}\dfrac{dT}{dr} = 0$ can be written as $\dfrac{dS}{dr} + \dfrac{2}{r}S = 0$. Thus,

$$\frac{dS}{dr} = \frac{-2S}{r} \quad\Rightarrow\quad \frac{dS}{S} = -\frac{2}{r}\,dr \quad\Rightarrow\quad \int \frac{1}{S}\,dS = \int -\frac{2}{r}\,dr \quad\Rightarrow\quad \ln|S| = -2\ln|r| + C.$$ Assuming $S = dT/dr > 0$

and $r > 0$, we have $S = e^{-2\ln r + C} = e^{\ln r^{-2}}e^C = r^{-2}k \quad [k = e^C] \quad\Rightarrow\quad S = \dfrac{1}{r^2}k \quad\Rightarrow\quad \dfrac{dT}{dr} = \dfrac{1}{r^2}k \quad\Rightarrow$

$$dT = \frac{1}{r^2}k\,dr \quad\Rightarrow\quad \int dT = \int \frac{1}{r^2}k\,dr \quad\Rightarrow\quad T(r) = -\frac{k}{r} + A.$$

$T(1) = 15 \quad\Rightarrow\quad 15 = -k + A$ **(1)** and $T(2) = 25 \quad\Rightarrow\quad 25 = -\frac{1}{2}k + A$ **(2)**.

Now solve for k and A: -2**(2)** + **(1)** $\quad\Rightarrow\quad -35 = -A$, so $A = 35$ and $k = 20$, and $T(r) = -20/r + 35$.

48. The air in a room with volume 180 m^3 contains 0.15% carbon dioxide initially. Fresher air with only 0.05% carbon dioxide flows into the room at a rate of 2 m^3/min and the mixed air flows out at the same rate. Find the percentage of carbon dioxide in the room as a function of time. What happens in the long run?

Solution:

Let $y(t)$ be the amount of carbon dioxide in the room after t minutes. Then $y(0) = 0.0015(180) = 0.27$ m^3. The amount of air in the room is 180 m^3 at all times, so the percentage at time t (in mimutes) is $y(t)/180 \times 100$, and the change in the amount of carbon dioxide with respect to time is

$$\frac{dy}{dt} = (0.0005)\left(2\,\frac{\text{m}^3}{\text{min}}\right) - \frac{y(t)}{180}\left(2\,\frac{\text{m}^3}{\text{min}}\right) = 0.001 - \frac{y}{90} = \frac{9-100y}{9000}\,\frac{\text{m}^3}{\text{min}}$$

Hence, $\displaystyle\int \frac{dy}{9-100y} = \int \frac{dt}{9000}$ and $-\dfrac{1}{100}\ln|9-100y| = \dfrac{1}{9000}t + C$. Because $y(0) = 0.27$, we have $-\frac{1}{100}\ln 18 = C$, so $-\frac{1}{100}\ln|9-100y| = \frac{1}{9000}t - \frac{1}{100}\ln 18 \;\Rightarrow\; \ln|9-100y| = -\frac{1}{90}t + \ln 18 \;\Rightarrow$ $\ln|9-100y| = \ln e^{-t/90} + \ln 18 \;\Rightarrow\; \ln|9-100y| = \ln(18e^{-t/90})$, and $|9-100y| = 18e^{-t/90}$. Since y is continuous, $y(0) = 0.27$, and the right-hand side is never zero, we deduce that $9-100y$ is always negative. Thus, $|9-100y| = 100y-9$ and we have $100y - 9 = 18e^{-t/90} \;\Rightarrow\; 100y = 9 + 18e^{-t/90} \;\Rightarrow\; y = 0.09 + 0.18e^{-t/90}$. The percentage of carbon dioxide in the room is

$$p(t) = \frac{y}{180} \times 100 = \frac{0.09 + 0.18e^{-t/90}}{180} \times 100 = (0.0005 + 0.001e^{-t/90}) \times 100 = 0.05 + 0.1e^{-t/90}$$

In the long run, we have $\lim\limits_{t\to\infty} p(t) = 0.05 + 0.1(0) = 0.05$; that is, the amount of carbon dioxide approaches 0.05% as time goes on.

54. A model for tumor growth is given by the Gompertz equation

$$\frac{dV}{dt} = a(\ln b - \ln V)V$$

where a and b are positive constants and V is the volume of the tumor measured in mm^3.
(a) Find a family of solutions for tumor volume as a function of time.
(b) Find the solution that has an initial tumor volume of $V(0) = 1$ mm^3.

Solution:

(a) $\dfrac{dV}{dt} = a(\ln b - \ln V)V \;\Rightarrow\; \dfrac{dV}{dt} = -aV(\ln V - \ln b) \;\Rightarrow\; \dfrac{dV}{V\ln(V/b)} = -a\,dt \;\Rightarrow$

$\displaystyle\int \frac{dV}{V\ln(V/b)} = \int -a\,dt \;\Rightarrow\; \int \frac{1}{u}\,du = \int -a\,dt \quad \begin{bmatrix} u = \ln(V/b), \\ du = (1/V)\,dV \end{bmatrix} \;\Rightarrow\; \ln|u| = -at + k \;\Rightarrow$

$|u| = e^{-at}e^k \;\Rightarrow\; u = Ce^{-at}$ [where $C = \pm e^k$] $\;\Rightarrow\; \ln(V/b) = Ce^{-at} \;\Rightarrow\; \dfrac{V}{b} = e^{Ce^{-at}} \;\Rightarrow$

$V = be^{Ce^{-at}}$ with $C \neq 0$.

(b) $V(0) = 1 \;\Rightarrow\; 1 = be^{Ce^{-a(0)}} \;\Rightarrow\; 1 = be^C \;\Rightarrow\; b = e^{-C}$, so $V = e^{-C}e^{Ce^{-at}} = e^{Ce^{-at}-C} = e^{C(e^{-at}-1)}$.