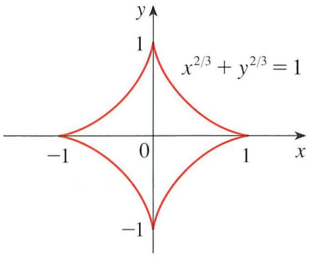
Section 8.1 Arc Length

39. Find the length of the astroid $x^{2/3} + y^{2/3} = 1$.



Solution:

The astroid $x^{2/3} + y^{2/3} = 1$ has an equal length of arc in each quadrant. Thus, we can find the length of the curve in the first quadrant and then multiply by 4. The top half of the astroid has equation $y = (1 - x^{2/3})^{3/2}$. Then

$dy/dx = -x^{-1/3}(1 - x^{2/3})^{1/2} \quad \Rightarrow \quad 1 + (dy/dx)^2 = 1 + \left[-x^{-1/3}(1 - x^{2/3})^{1/2}\right]^2 = 1 + x^{-2/3}(1 - x^{2/3}) = x^{-2/3}$.

So the portion of the astroid in quadrant 1 has length $L = \int_0^1 \sqrt{x^{-2/3}}\, dx = \int_0^1 x^{-1/3}\, dx = \left[\frac{3}{2}x^{2/3}\right]_0^1 = \frac{3}{2} - 0 = \frac{3}{2}$. Thus, the astroid has length $4\left(\frac{3}{2}\right) = 6$.

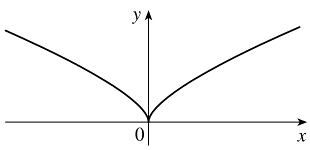
40. (a) Sketch the curve $y^3 = x^2$.

(b) Use Formulas 3 and 4 to set up two integrals for the arc length from $(0,0)$ to $(1,1)$. Observe that one of these is an improper integral and evaluate both of them.

(c) Find the length of the arc of this curve from $(-1,1)$ to $(8,4)$.

Solution:

(a)



(b) $y = x^{2/3} \quad \Rightarrow \quad 1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + \left(\frac{2}{3}x^{-1/3}\right)^2 = 1 + \frac{4}{9}x^{-2/3}$. So $L = \int_0^1 \sqrt{1 + \frac{4}{9}x^{-2/3}}\, dx$ [an improper integral].

$x = y^{3/2} \quad \Rightarrow \quad 1 + \left(\dfrac{dx}{dy}\right)^2 = 1 + \left(\frac{3}{2}y^{1/2}\right)^2 = 1 + \frac{9}{4}y$. So $L = \int_0^1 \sqrt{1 + \frac{9}{4}y}\, dy$.

The second integral equals $\frac{4}{9} \cdot \frac{2}{3}\left[\left(1 + \frac{9}{4}y\right)^{3/2}\right]_0^1 = \frac{8}{27}\left(\frac{13\sqrt{13}}{8} - 1\right) = \frac{13\sqrt{13} - 8}{27}$.

The first integral can be evaluated as follows:

$$\int_0^1 \sqrt{1 + \tfrac{4}{9}x^{-2/3}}\, dx = \lim_{t \to 0^+} \int_t^1 \frac{\sqrt{9x^{2/3} + 4}}{3x^{1/3}}\, dx = \lim_{t \to 0^+} \int_{9t^{2/3}}^9 \frac{\sqrt{u+4}}{18}\, du \qquad \left[\begin{matrix} u = 9x^{2/3}, \\ du = 6x^{-1/3}\, dx \end{matrix}\right]$$

$$= \int_0^9 \frac{\sqrt{u+4}}{18}\, du = \frac{1}{18} \cdot \left[\frac{2}{3}(u+4)^{3/2}\right]_0^9 = \frac{1}{27}(13^{3/2} - 4^{3/2}) = \frac{13\sqrt{13} - 8}{27}$$

(c) L = length of the arc of this curve from $(-1,1)$ to $(8,4)$

$$= \int_0^1 \sqrt{1 + \frac{9}{4}y}\, dy + \int_0^4 \sqrt{1 + \frac{9}{4}y}\, dy = \frac{13\sqrt{13} - 8}{27} + \frac{8}{27}\left[\left(1 + \frac{9}{4}y\right)^{3/2}\right]_0^4 \qquad \text{[from part (b)]}$$

$$= \frac{13\sqrt{13} - 8}{27} + \frac{8}{27}(10\sqrt{10} - 1) = \frac{13\sqrt{13} + 80\sqrt{10} - 16}{27}$$

43. Find the arc length function for the curve $y = \sin^{-1} x + \sqrt{1-x^2}$ with starting point $(0, 1)$.

Solution:

$$y = \sin^{-1} x + \sqrt{1-x^2} \quad \Rightarrow \quad y' = \frac{1}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} = \frac{1-x}{\sqrt{1-x^2}} \quad \Rightarrow$$

$$1 + (y')^2 = 1 + \frac{(1-x)^2}{1-x^2} = \frac{1-x^2+1-2x+x^2}{1-x^2} = \frac{2-2x}{1-x^2} = \frac{2(1-x)}{(1+x)(1-x)} = \frac{2}{1+x} \quad \Rightarrow$$

$\sqrt{1+(y')^2} = \sqrt{\dfrac{2}{1+x}}$. Thus, the arc length function with starting point $(0, 1)$ is given by

$$s(x) = \int_0^x \sqrt{1+[f'(t)]^2}\,dt = \int_0^x \sqrt{\frac{2}{1+t}}\,dt = \sqrt{2}\left[2\sqrt{1+t}\,\right]_0^x = 2\sqrt{2}\left(\sqrt{1+x}-1\right).$$

46. A steady wind blows a kite due west. The kite's height above ground from horizontal position $x = 0$ to $x = 25$ m is given by $y = 50 - 0.1(x-15)^2$. Find the distance traveled by the kite.

Solution:

$y = 50 - \frac{1}{10}(x-15)^2 \Rightarrow y' = -\frac{1}{5}(x-15) \Rightarrow 1 + (y')^2 = 1 + \frac{1}{5^2}(x-15)^2$, so the distance traveled by the kite is

$$L = \int_0^{25} \sqrt{1 + \frac{1}{5^2}(x-15)^2}\,dx = \int_{-3}^{2} \sqrt{1+u^2}\,(5\,du) \quad \begin{bmatrix} u = \frac{1}{5}(x-15), \\ du = \frac{1}{5}dx \end{bmatrix}$$

$$\overset{21}{=} 5\left[\frac{1}{2}u\sqrt{1+u^2} + \frac{1}{2}\ln\left(u + \sqrt{1+u^2}\right)\right]_{-3}^{2} = \frac{5}{2}\left[2\sqrt{5} + \ln\left(2+\sqrt{5}\right) + 3\sqrt{10} - \ln\left(-3+\sqrt{10}\right)\right]$$

$$\approx 43.05 \text{ m}$$