Section 7.4 Integration of Rational Functions by Partial Fractions

27. Evaluate the integral. $\int \frac{4x}{x^3+x^2+x+1}dx$

Solution:

$\dfrac{4x}{x^3+x^2+x+1} = \dfrac{4x}{x^2(x+1)+1(x+1)} = \dfrac{4x}{(x+1)(x^2+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+1}$. Multiply both sides by $(x+1)(x^2+1)$ to get $4x = A(x^2+1) + (Bx+C)(x+1) \quad \Leftrightarrow \quad 4x = Ax^2 + A + Bx^2 + Bx + Cx + C \quad \Leftrightarrow$ $4x = (A+B)x^2 + (B+C)x + (A+C)$. Comparing coefficients gives us the following system of equations:

$$A + B = 0 \quad \textbf{(1)} \qquad\qquad B + C = 4 \quad \textbf{(2)} \qquad\qquad A + C = 0 \quad \textbf{(3)}$$

Subtracting equation **(1)** from equation **(2)** gives us $-A + C = 4$, and adding that equation to equation **(3)** gives us $2C = 4 \quad \Leftrightarrow \quad C = 2$, and hence $A = -2$ and $B = 2$. Thus,

$$\int \frac{4x}{x^3+x^2+x+1}\,dx = \int \left(\frac{-2}{x+1} + \frac{2x+2}{x^2+1} \right) dx = \int \left(\frac{-2}{x+1} + \frac{2x}{x^2+1} + \frac{2}{x^2+1} \right) dx$$
$$= -2\ln|x+1| + \ln(x^2+1) + 2\tan^{-1} x + C$$

50. Make a substitution to express the integrand as a rational function and then evaluate the integral. $\int \frac{\sqrt{1+\sqrt{x}}}{x}dx$

Solution:

Let $u = \sqrt{1+\sqrt{x}}$, so that $u^2 = 1 + \sqrt{x}$, $x = (u^2-1)^2$, and $dx = 2(u^2-1)\cdot 2u\,du = 4u(u^2-1)\,du$. Then $\displaystyle\int \frac{\sqrt{1+\sqrt{x}}}{x}\,dx = \int \frac{u}{(u^2-1)^2}\cdot 4u(u^2-1)\,du = \int \frac{4u^2}{u^2-1}\,du = \int \left(4 + \frac{4}{u^2-1}\right) du$. Now $\dfrac{4}{u^2-1} = \dfrac{A}{u+1} + \dfrac{B}{u-1} \quad \Rightarrow \quad 4 = A(u-1) + B(u+1)$. Setting $u = 1$ gives $4 = 2B$, so $B = 2$. Setting $u = -1$ gives $4 = -2A$, so $A = -2$. Thus,

$$\int \left(4 + \frac{4}{u^2-1}\right) du = \int \left(4 - \frac{2}{u+1} + \frac{2}{u-1}\right) du = 4u - 2\ln|u+1| + 2\ln|u-1| + C$$
$$= 4\sqrt{1+\sqrt{x}} - 2\ln\left(\sqrt{1+\sqrt{x}} + 1\right) + 2\ln\left(\sqrt{1+\sqrt{x}} - 1\right) + C$$

58. Use integration by parts, together with the techniques of this section, to evaluate the integral. $\int x\tan^{-1} x dx$

Solution:

Let $u = \tan^{-1} x$, $dv = x\,dx \quad \Rightarrow \quad du = dx/(1+x^2)$, $v = \frac{1}{2}x^2$.

Then $\displaystyle\int x\tan^{-1} x\,dx = \tfrac{1}{2}x^2\tan^{-1} x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx$. To evaluate the last integral, use long division or observe that $\displaystyle\int \frac{x^2}{1+x^2}\,dx = \int \frac{(1+x^2)-1}{1+x^2}\,dx = \int 1\,dx - \int \frac{1}{1+x^2}\,dx = x - \tan^{-1} x + C_1$. So $\int x\tan^{-1} x\,dx = \frac{1}{2}x^2\tan^{-1} x - \frac{1}{2}(x - \tan^{-1} x + C_1) = \frac{1}{2}(x^2\tan^{-1} x + \tan^{-1} x - x) + C$.

60. Evaluate $\int \frac{1}{x^2+k}dx$ by considering several cases for the constant k.

Solution:

$\boldsymbol{k=0}$: $$\int \frac{dx}{x^2+k} = \int \frac{dx}{x^2} = -\frac{1}{x} + C$$

$\boldsymbol{k>0}$: $$\int \frac{dx}{x^2+k} = \int \frac{dx}{x^2+(\sqrt{k})^2} = \frac{1}{\sqrt{k}} \tan^{-1}\left(\frac{x}{\sqrt{k}}\right) + C$$

$\boldsymbol{k<0}$: $$\int \frac{dx}{x^2+k} = \int \frac{dx}{x^2-(-k)} = \int \frac{dx}{x^2-\left(\sqrt{-k}\right)^2} = \frac{1}{2\sqrt{-k}} \ln\left|\frac{x-\sqrt{-k}}{x+\sqrt{-k}}\right| + C$$