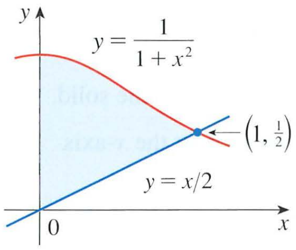
Section 6.3 Volumes by Cylindrical Shells

47. A solid is obtained by rotating the shaded region about the specified line.

(a) Set up an integral using any method to find the volume of the solid.

(b) Evaluate the integral to find the volume of the solid.

About the y-axis



Solution:

(a) Use shells. Each shell has radius x, circumference $2\pi x$, and height $\dfrac{1}{1+x^2} - \dfrac{x}{2}$.

$$V = \int_0^1 2\pi x\left(\frac{1}{1+x^2} - \frac{x}{2}\right) dx.$$

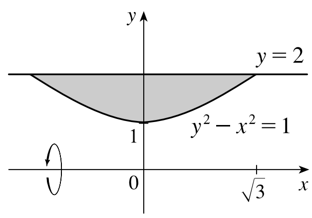
(b) $$V = \int_0^1 2\pi x\left(\frac{1}{1+x^2} - \frac{x}{2}\right) dx = \int_0^1 2\pi\left(\frac{x}{1+x^2} - \frac{1}{2}x^2\right) dx = 2\pi\left[\frac{1}{2}\ln\left|1+x^2\right| - \frac{1}{6}x^3\right]_0^1$$

$$= 2\pi\left[\left(\frac{1}{2}\ln 2 - \frac{1}{6}\right) - 0\right] = \pi\left(\ln 2 - \frac{1}{3}\right)$$

55. The region bounded by the given curves is rotated about the specified axis. Find the volume of the resulting solid by any method. $y^2 - x^2 = 1$, $y = 2$; about x-axis.

Solution:

Use washers: $y^2 - x^2 = 1 \quad \Rightarrow \quad y = \pm\sqrt{x^2 \pm 1}$



$$V = \int_{-\sqrt{3}}^{\sqrt{3}} \pi\left[(2-0)^2 - \left(\sqrt{x^2+1} - 0\right)^2\right] dx$$

$$= 2\pi\int_0^{\sqrt{3}} [4 - (x^2+1)]\, dx \qquad \text{[by symmetry]}$$

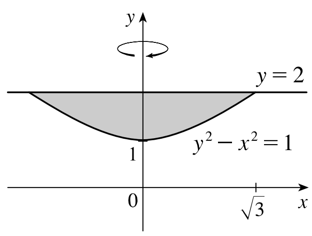
$$= 2\pi\int_0^{\sqrt{3}} (3 - x^2)\, dx = 2\pi\left[3x - \tfrac{1}{3}x^3\right]_0^{\sqrt{3}}$$

$$= 2\pi\left(3\sqrt{3} - \sqrt{3}\right) = 4\sqrt{3}\,\pi$$

56. The region bounded by the given curves is rotated about the specified axis. Find the volume of the resulting solid by any method. $y^2 - x^2 = 1$, $y = 2$; about y-axis.

Solution:

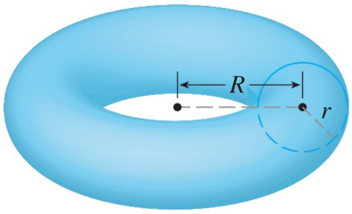
Use disks: $y^2 - x^2 = 1 \quad \Rightarrow \quad x = \pm\sqrt{y^2 - 1}$



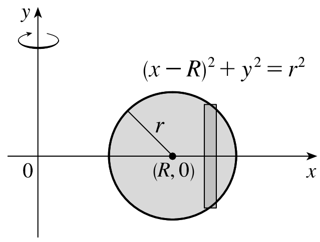
$$V = \pi\int_1^2 \left(\sqrt{y^2-1}\right)^2 dy = \pi\int_1^2 (y^2 - 1)\, dy$$

$$= \pi\left[\tfrac{1}{3}y^3 - y\right]_1^2 = \pi\left[\left(\tfrac{8}{3} - 2\right) - \left(\tfrac{1}{3} - 1\right)\right] = \tfrac{4}{3}\pi$$

62. Use cylindrical shells to find the volume of the solid. The solid torus of Exercise 6.2.75



Solution:



$V = \int_{R-r}^{R+r} 2\pi x \cdot 2\sqrt{r^2 - (x-R)^2}\, dx$

$= \int_{-r}^{r} 4\pi(u+R)\sqrt{r^2 - u^2}\, du \qquad$ [let $u = x - R$]

$= 4\pi R \int_{-r}^{r} \sqrt{r^2 - u^2}\, du + 4\pi \int_{-r}^{r} u\sqrt{r^2 - u^2}\, du$

The first integral is the area of a semicircle of radius r, that is, $\frac{1}{2}\pi r^2$, and the second is zero since the integrand is an odd function. Thus, $V = 4\pi R\left(\frac{1}{2}\pi r^2\right) + 4\pi \cdot 0 = 2\pi^2 R r^2$.