Section 6.2 Volume

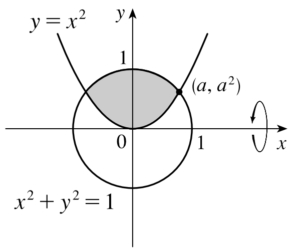
44. Set up an integral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. Then use your calculator to evaluate the integral correct to five decimal places.

$y = x^2$, $x^2 + y^2 = 1$, $y \geq 0$

(a) About the x-axis (b) About the y-axis.

Solution:

(a) About the x-axis:



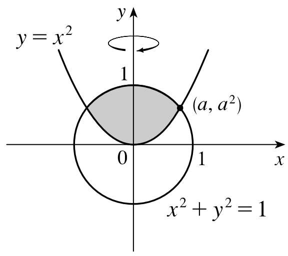
$$y = x^2 \text{ and } x^2 + y^2 = 1 \quad \Rightarrow \quad x^2 + x^4 = 1 \quad \Rightarrow \quad x^4 + x^2 - 1 = 0 \quad \Rightarrow$$

$$x^2 = \frac{-1 + \sqrt{5}}{2} \approx 0.618 \quad \Rightarrow \quad x = \pm a = \pm\sqrt{\frac{-1+\sqrt{5}}{2}} \approx \pm 0.786.$$

$$V = \int_{-a}^{a} \pi\left[\left(\sqrt{1-x^2}\right)^2 - (x^2)^2\right] dx = 2\pi \int_0^a (1 - x^2 - x^4)\, dx$$

$$\approx 3.54459$$

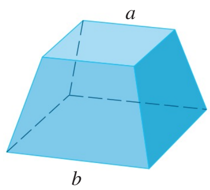
(b) About the y-axis:



$$V = \int_0^{a^2} \pi\left(\sqrt{y}\right)^2 dy + \int_{a^2}^{1} \pi\left(\sqrt{1-y^2}\right)^2 dy$$

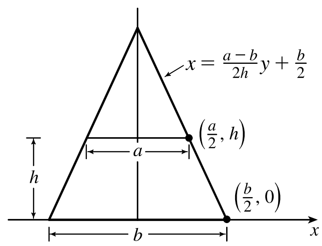
$$= \pi \int_0^{a^2} y\, dy + \pi \int_{a^2}^{1} (1 - y^2)\, dy \approx 0.99998$$

62. A frustum of a pyramid with square base of side b, square top of side a, and height h. What happens if $a = b$? What happens if $a = 0$?



Solution:

An equation of the line is $x = \dfrac{\Delta x}{\Delta y}\, y + (x\text{-intercept}) = \dfrac{a/2 - b/2}{h - 0}\, y + \dfrac{b}{2} = \dfrac{a-b}{2h}\, y + \dfrac{b}{2}$.



$$V = \int_0^h A(y)\, dy = \int_0^h (2x)^2\, dy$$

$$= \int_0^h \left[2\left(\frac{a-b}{2h}y + \frac{b}{2}\right)\right]^2 dy = \int_0^h \left[\frac{a-b}{h}y + b\right]^2 dy$$

$$= \int_0^h \left[\frac{(a-b)^2}{h^2}y^2 + \frac{2b(a-b)}{h}y + b^2\right] dy$$

$$= \left[\frac{(a-b)^2}{3h^2}y^3 + \frac{b(a-b)}{h}y^2 + b^2 y\right]_0^h$$

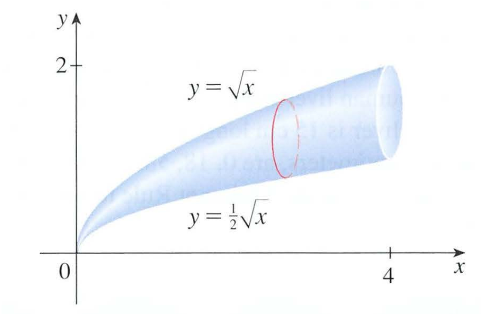
$$= \tfrac{1}{3}(a-b)^2 h + b(a-b)h + b^2 h = \tfrac{1}{3}\left(a^2 - 2ab + b^2 + 3ab\right)h$$

$$= \tfrac{1}{3}\left(a^2 + ab + b^2\right)h$$

[Note that this can be written as $\frac{1}{3}\left(A_1 + A_2 + \sqrt{A_1 A_2}\,\right)h$, as in Exercise 48.]

If $a = b$, we get a rectangular solid with volume $b^2 h$. If $a = 0$, we get a square pyramid with volume $\frac{1}{3}b^2 h$.

74. Cross-sections of the solid S in planes perpendicular to the x-axis are circles with diameters extending from the curve $y = \frac{1}{2}\sqrt{x}$ to the curve $y = \sqrt{x}$ for $0 \le x \le 4$.



Solution:

The cross-section of S at coordinate x, $0 \le x \le 4$, is a circle centered at the point $\left(x, \frac{1}{2}\left(\frac{1}{2}\sqrt{x} + \sqrt{x}\right)\right)$ with radius $\frac{1}{2}\left(\sqrt{x} - \frac{1}{2}\sqrt{x}\right)$. The area of the cross-section is $A(x) = \pi\left[\frac{1}{2}\left(\sqrt{x} - \frac{1}{2}\sqrt{x}\right)\right]^2 = \pi \cdot \frac{1}{4} \cdot \left(\frac{1}{2}\sqrt{x}\right)^2 = \frac{\pi x}{16}$. The volume of S is $V = \int_0^4 A(x)\,dx = \int_0^4 \frac{\pi x}{16}\,dx = \frac{\pi}{32}\left[x^2\right]_0^4 = \frac{\pi}{32}(16 - 0) = \frac{\pi}{2}$.

86. Suppose that a region $\mathfrak{R}$ has area A and lies above the x-axis. When $\mathfrak{R}$ is rotated about the x-axis, it sweeps out a solid with volume V_1. When $\mathfrak{R}$ is rotated about the line $y = -k$ (where k is a positive number), it sweeps out a solid with volume V_2. Express V_2 in terms of V_1, k, and A.

Solution:

It suffices to consider the case where $\mathscr{R}$ is bounded by the curves $y = f(x)$ and $y = g(x)$ for $a \le x \le b$, where $g(x) \le f(x)$ for all x in $[a, b]$, since other regions can be decomposed into subregions of this type. We are concerned with the volume obtained when $\mathscr{R}$ is rotated about the line $y = -k$, which is equal to

$$\begin{aligned} V_2 &= \pi \int_a^b \left([f(x) + k]^2 - [g(x) + k]^2\right) dx \\ &= \pi \int_a^b \left([f(x)]^2 - [g(x)]^2\right) dx + 2\pi k \int_a^b [f(x) - g(x)]\,dx = V_1 + 2\pi k A \end{aligned}$$