Section 5.5 The Substitution Rule

78. Evaluate the definite integral $\int_1^4 \frac{1}{(x+1)\sqrt{x}} dx$.

Solution:

Let $u = \sqrt{x}$. Then $u^2 = x$, $2u\,du = dx$, $du = \dfrac{1}{2\sqrt{x}}\,dx$, and $\dfrac{1}{\sqrt{x}}\,dx = 2\,du$. When $x = 1$, $u = 1$; when $x = 4$, $u = 2$.

Thus,

$$\int_1^4 \frac{1}{(x+1)\sqrt{x}}\,dx = \int_1^2 \frac{1}{u^2+1}\,(2\,du) = 2\Big[\arctan u\Big]_1^2 = 2(\arctan 2 - \arctan 1)$$
$$= 2\Big(\arctan 2 - \frac{\pi}{4}\Big) = 2\arctan 2 - \frac{\pi}{2}$$

80. Evaluate the definite integral $\int_1^{16} \frac{x^{1/2}}{1+x^{3/4}} dx$.

Solution:

Let $u = 1 + x^{3/4}$. Then $x^{3/4} = u - 1$, $du = \frac{3}{4}x^{-1/4}\,dx$, and $x^{-1/4}\,dx = \frac{4}{3}\,du$. When $x = 1$, $u = 2$; when $x = 16$, $u = 9$.

Thus,

$$\int_1^{16} \frac{x^{1/2}}{1+x^{3/4}}\,dx = \int_1^{16} \frac{x^{3/4}\cdot x^{-1/4}}{1+x^{3/4}}\,dx = \int_2^9 \frac{u-1}{u}\left(\frac{4}{3}\,du\right) = \frac{4}{3}\int_2^9 \left(1 - \frac{1}{u}\right)du = \frac{4}{3}\Big[u - \ln|u|\Big]_2^9$$
$$= \tfrac{4}{3}[(9-\ln 9) - (2 - \ln 2)] = \tfrac{4}{3}(7 - \ln 9 + \ln 2) = \tfrac{4}{3}\left(7 + \ln\tfrac{2}{9}\right)$$

83. Evaluate $\int_{-2}^2 (x+3)\sqrt{4-x^2}dx$ by writing it as a sum of two integrals and interpreting one of those integrals in terms of an area.

Solution:

First write the integral as a sum of two integrals:

$I = \int_{-2}^2 (x+3)\sqrt{4-x^2}\,dx = I_1 + I_2 = \int_{-2}^2 x\sqrt{4-x^2}\,dx + \int_{-2}^2 3\sqrt{4-x^2}\,dx$. $I_1 = 0$ by Theorem 7(b), since $f(x) = x\sqrt{4-x^2}$ is an odd function and we are integrating from $x = -2$ to $x = 2$. We interpret I_2 as three times the area of a semicircle with radius 2, so $I = 0 + 3\cdot\frac{1}{2}\left(\pi\cdot 2^2\right) = 6\pi$.

94. If f is continuous and $\int_0^9 f(x)dx = 4$, find $\int_0^3 xf(x^2)dx$.

Solution:

Let $u = x^2$. Then $du = 2x\,dx$, so $\int_0^3 xf(x^2)\,dx = \int_0^9 f(u)\left(\frac{1}{2}\,du\right) = \frac{1}{2}\int_0^9 f(u)\,du = \frac{1}{2}(4) = 2$.

98. If f is continuous on $[0,\pi]$, use the substitution $u = \pi - x$ to show that

$$\int_0^\pi xf(\sin x)dx = \frac{\pi}{2}\int_0^\pi f(\sin x)dx$$

Solution:

Let $u = \pi - x$. Then $du = -dx$. When $x = \pi$, $u = 0$ and when $x = 0$, $u = \pi$. So

$$\int_0^\pi xf(\sin x)\,dx = -\int_\pi^0 (\pi - u)\,f(\sin(\pi - u))\,du = \int_0^\pi (\pi - u)\,f(\sin u)\,du$$
$$= \pi\int_0^\pi f(\sin u)\,du - \int_0^\pi u\,f(\sin u)\,du = \pi\int_0^\pi f(\sin x)\,dx - \int_0^\pi x\,f(\sin x)\,dx \quad \Rightarrow$$

$2\int_0^\pi x\,f(\sin x)\,dx = \pi\int_0^\pi f(\sin x)\,dx \quad \Rightarrow \quad \int_0^\pi x\,f(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx$.

99. Use Exercise 98 to evaluate the integral $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$.

Solution:

$\dfrac{x \sin x}{1+\cos^2 x} = x \cdot \dfrac{\sin x}{2-\sin^2 x} = x\, f(\sin x)$, where $f(t) = \dfrac{t}{2-t^2}$. By Exercise 92,

$$\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx = \int_0^{\pi} x\, f(\sin x)\, dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\, dx = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\, dx$$

Let $u = \cos x$. Then $du = -\sin x\, dx$. When $x = \pi$, $u = -1$ and when $x = 0$, $u = 1$. So

$$\frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\, dx = -\frac{\pi}{2}\int_1^{-1} \frac{du}{1+u^2} = \frac{\pi}{2}\int_{-1}^{1} \frac{du}{1+u^2} = \frac{\pi}{2}\left[\tan^{-1} u\right]_{-1}^{1}$$

$$= \frac{\pi}{2}[\tan^{-1} 1 - \tan^{-1}(-1)] = \frac{\pi}{2}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \frac{\pi^2}{4}$$