Section 5.3 The Fundamental Theorem of Calculus

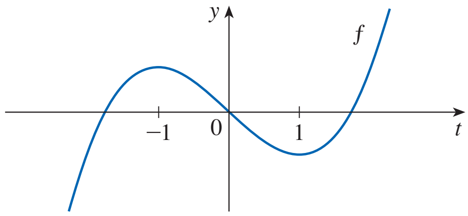
68. Find the derivative of the function. $g(x)=\int_{1-2x}^{1+2x} t\sin t dt$.

Solution:

$$g(x)=\int_{1-2x}^{1+2x} t\sin t\,dt=\int_{1-2x}^{0} t\sin t\,dt+\int_{0}^{1+2x} t\sin t\,dt=-\int_{0}^{1-2x} t\sin t\,dt+\int_{0}^{1+2x} t\sin t\,dt \quad\Rightarrow$$

$$g'(x)=-(1-2x)\sin(1-2x)\cdot\frac{d}{dx}(1-2x)+(1+2x)\sin(1+2x)\cdot\frac{d}{dx}(1+2x)$$
$$=2(1-2x)\sin(1-2x)+2(1+2x)\sin(1+2x)$$

74. Let $F(x)=\int_1^x f(t)dt$, where f is the function whose graph is shown. Where is F concave downward?



Solution:

If $F(x)=\int_1^x f(t)\,dt$, then by FTC1, $F'(x)=f(x)$, and also, $F''(x)=f'(x)$. F is concave downward where F'' is negative; that is, where f' is negative. The given graph shows that f is decreasing ($f'<0$) on the interval $(-1,1)$.

76. If $f(x)=\int_0^{\sin x}\sqrt{1+t^2}dt$ and $g(y)=\int_3^y f(x)dx$, find $g''(\frac{\pi}{6})$.

Solution:

$g(y)=\int_3^y f(x)\,dx \quad\Rightarrow\quad g'(y)=f(y)$. Since $f(x)=\int_0^{\sin x}\sqrt{1+t^2}\,dt$, $g''(y)=f'(y)=\sqrt{1+\sin^2 y}\cdot\cos y$,

so $g''(\frac{\pi}{6})=\sqrt{1+\sin^2(\frac{\pi}{6})}\cdot\cos\frac{\pi}{6}=\sqrt{1+(\frac{1}{2})^2}\cdot\frac{\sqrt{3}}{2}=\frac{\sqrt{5}}{2}\cdot\frac{\sqrt{3}}{2}=\frac{\sqrt{15}}{4}$.

78. Use l'Hospital's Rule to evaluate the limit. $\lim\limits_{x\to\infty}\frac{1}{x^2}\int_0^x\ln(1+e^t)dt$.

Solution:

$$\lim_{x\to\infty}\frac{1}{x^2}\int_0^x\ln(1+e^t)\,dt=\lim_{x\to\infty}\frac{\int_0^x\ln(1+e^t)\,dt}{x^2}\quad\left[\text{form }\frac{\infty}{\infty}\right]\quad\overset{\text{H}}{=}\lim_{x\to\infty}\frac{\ln(1+e^x)}{2x}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{\frac{e^x}{1+e^x}}{2}$$
$$=\lim_{x\to\infty}\frac{e^x}{2(1+e^x)}=\lim_{x\to\infty}\frac{e^x/e^x}{2(1+e^x)/e^x}=\lim_{x\to\infty}\frac{1}{2\left(\frac{1}{e^x}+1\right)}=\frac{1}{2(0+1)}=\frac{1}{2}$$

93. Find a function f and a number a such that

$$6+\int_a^x\frac{f(t)}{t^2}dt=2\sqrt{x}\quad\text{for all } x>0$$

Solution:

Using FTC1, we differentiate both sides of $6+\int_a^x\frac{f(t)}{t^2}\,dt=2\sqrt{x}$ to get $\frac{f(x)}{x^2}=2\frac{1}{2\sqrt{x}}\quad\Rightarrow\quad f(x)=x^{3/2}$.

To find a, we substitute $x=a$ in the original equation to obtain $6+\int_a^a\frac{f(t)}{t^2}\,dt=2\sqrt{a}\quad\Rightarrow\quad 6+0=2\sqrt{a}\quad\Rightarrow$

$3=\sqrt{a}\quad\Rightarrow\quad a=9$.