Section 5.1 The Area and Distance Problems

18. Use Definition 2 to find an expression for the area under the graph of f as a limit. Do not evaluate the limit. $f(x) = x + \ln x$, $3 \le x \le 8$.

Solution:

$f(x) = x + \ln x,\ 3 \le x \le 8.$ $\Delta x = (8-3)/n = 5/n$ and $x_i = 3 + i\,\Delta x = 3 + 5i/n$.

$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} [(3+5i/n) + \ln(3+5i/n)] \cdot \dfrac{5}{n}.$

22. Determine a region whose area is equal to the given limit. Do not evaluate the limit. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\sqrt{1+\frac{3i}{n}}$.

Solution:

$\lim_{n\to\infty} \sum_{i=1}^{n} \dfrac{3}{n}\sqrt{1+\dfrac{3i}{n}}$ can be interpreted as the area of the region lying under the graph of $y = \sqrt{1+x}$ on the interval $[0,3]$, since for $y = \sqrt{1+x}$ on $[0,3]$ with $\Delta x = \dfrac{3-0}{n} = \dfrac{3}{n}$, $x_i = 0 + i\,\Delta x = \dfrac{3i}{n}$, and $x_i^* = x_i$, the expression for the area is $A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1+\dfrac{3i}{n}}\,\dfrac{3}{n}$. Note that this answer is not unique. We could use $y = \sqrt{x}$ on $[1,4]$ or, in general, $y = \sqrt{x-n}$ on $[n+1, n+4]$, where n is any real number.

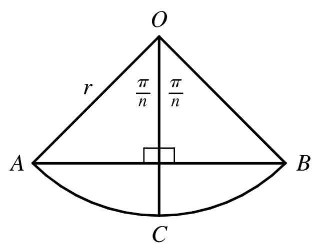
34. (a) Let A_n be the area of a polygon with n equal sides inscribed in a circle with radius r. By dividing the polygon into n congruent triangles with central angle $2\pi/n$, show that

$$A_n = \frac{1}{2} n r^2 \sin\frac{2\pi}{n}$$

(b) Show that $\lim_{n\to\infty} A_n = \pi r^2$. [Hint: Use Equation 3.3.5]

Solution:

(a)



The diagram shows one of the n congruent triangles, $\triangle AOB$, with central angle $2\pi/n$. O is the center of the circle and AB is one of the sides of the polygon. Radius OC is drawn so as to bisect $\angle AOB$. It follows that OC intersects AB at right angles and bisects AB. Thus, $\triangle AOB$ is divided into two right triangles with legs of length $\frac{1}{2}(AB) = r\sin(\pi/n)$ and $r\cos(\pi/n)$. $\triangle AOB$ has area $2 \cdot \frac{1}{2}[r\sin(\pi/n)][r\cos(\pi/n)] = r^2 \sin(\pi/n)\cos(\pi/n) = \frac{1}{2} r^2 \sin(2\pi/n)$, so $A_n = n \cdot \text{area}(\triangle AOB) = \frac{1}{2} n r^2 \sin(2\pi/n)$.

(b) To use Equation 3.3.2, $\lim_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$, we need to have the same expression in the denominator as we have in the argument of the sine function—in this case, $2\pi/n$.

$\lim_{n\to\infty} A_n = \lim_{n\to\infty} \frac{1}{2} n r^2 \sin(2\pi/n) = \lim_{n\to\infty} \frac{1}{2} n r^2 \dfrac{\sin(2\pi/n)}{2\pi/n} \cdot \dfrac{2\pi}{n} = \lim_{n\to\infty} \dfrac{\sin(2\pi/n)}{2\pi/n} \pi r^2$. Let $\theta = \dfrac{2\pi}{n}$.

Then as $n \to \infty$, $\theta \to 0$, so $\lim_{n\to\infty} \dfrac{\sin(2\pi/n)}{2\pi/n} \pi r^2 = \lim_{\theta\to 0} \dfrac{\sin\theta}{\theta} \pi r^2 = (1)\,\pi r^2 = \pi r^2$.