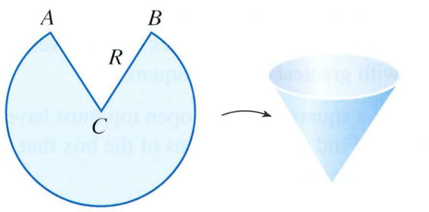
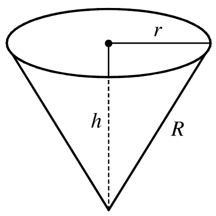
Section 4.7 Optimization Problems

47. A cone-shaped drinking cup is made from a circular piece of paper of radius R by cutting out a sector and joining the edges CA and CB. Find the maximum capacity of such a cup.



Solution:



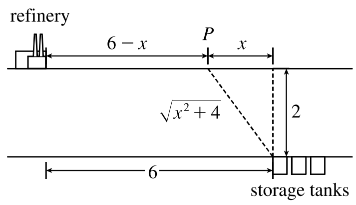
$h^2 + r^2 = R^2 \quad \Rightarrow \quad V = \frac{\pi}{3} r^2 h = \frac{\pi}{3}(R^2 - h^2)h = \frac{\pi}{3}(R^2 h - h^3)$.

$V'(h) = \frac{\pi}{3}(R^2 - 3h^2) = 0$ when $h = \frac{1}{\sqrt{3}}R$. This gives an absolute maximum, since $V'(h) > 0$ for $0 < h < \frac{1}{\sqrt{3}}R$ and $V'(h) < 0$ for $h > \frac{1}{\sqrt{3}}R$. The maximum volume is

$V\left(\frac{1}{\sqrt{3}}R\right) = \frac{\pi}{3}\left(\frac{1}{\sqrt{3}}R^3 - \frac{1}{3\sqrt{3}}R^3\right) = \frac{2}{9\sqrt{3}}\pi R^3$.

57. An oil refinery is located on the north bank of a straight river that is 2 km wide. A pipeline is to be constructed from the refinery to storage tanks located on the south bank of the river 6 km east of the refinery. The cost of laying pipe is $400, 000ykmoverlandtoapoint$P$onthenorthbankand$800,000ykm under the river to the tanks. To minimize the cost of the pipeline, where should P be located?

Solution:



There are $(6 - x)$ km over land and $\sqrt{x^2 + 4}$ km under the river.

We need to minimize the cost C (measured in \$100,000) of the pipeline.

$C(x) = (6 - x)(4) + \left(\sqrt{x^2 + 4}\right)(8) \quad \Rightarrow$

$C'(x) = -4 + 8 \cdot \frac{1}{2}(x^2 + 4)^{-1/2}(2x) = -4 + \dfrac{8x}{\sqrt{x^2 + 4}}$.

$C'(x) = 0 \quad \Rightarrow \quad 4 = \dfrac{8x}{\sqrt{x^2 + 4}} \quad \Rightarrow \quad \sqrt{x^2 + 4} = 2x \quad \Rightarrow \quad x^2 + 4 = 4x^2 \quad \Rightarrow \quad 4 = 3x^2 \quad \Rightarrow \quad x^2 = \frac{4}{3} \quad \Rightarrow$

$x = 2/\sqrt{3}$ $[0 \le x \le 6]$. Compare the costs for $x = 0, 2/\sqrt{3}$, and 6. $C(0) = 24 + 16 = 40$,

$C\left(2/\sqrt{3}\right) = 24 - 8/\sqrt{3} + 32/\sqrt{3} = 24 + 24/\sqrt{3} \approx 37.9$, and $C(6) = 0 + 8\sqrt{40} \approx 50.6$. So the minimum cost is about \$3.79 million when P is $6 - 2/\sqrt{3} \approx 4.85$ km east of the refinery.

65. (a) If $C(x)$ is the cost of producing x units of a commodity, then the average cost per unit is $c(x) = C(x)/x$. Show that if the average cost is a minimum, then the marginal cost equals the average cost.

(b) If $C(x) = 16000 + 200x + 4x^{3/2}$, in dollars, find (i) the cost, average cost, and marginal cost at a production level of 1000 units; (ii) the production level that will minimize the average cost; and (iii) the minimum average cost.

Solution:

(a) If $c(x) = \dfrac{C(x)}{x}$, then, by the Quotient Rule, we have $c'(x) = \dfrac{xC'(x) - C(x)}{x^2}$. Now $c'(x) = 0$ when $xC'(x) - C(x) = 0$ and this gives $C'(x) = \dfrac{C(x)}{x} = c(x)$. Therefore, the marginal cost equals the average cost.

(b) (i) $C(x) = 16{,}000 + 200x + 4x^{3/2}$, $C(1000) = 16{,}000 + 200{,}000 + 40{,}000\sqrt{10} \approx 216{,}000 + 126{,}491$, so $C(1000) \approx \$342{,}491$. $c(x) = C(x)/x = \dfrac{16{,}000}{x} + 200 + 4x^{1/2}$, $c(1000) \approx \$342.49/\text{unit}$. $C'(x) = 200 + 6x^{1/2}$, $C'(1000) = 200 + 60\sqrt{10} \approx \$389.74/\text{unit}$.

(ii) We must have $C'(x) = c(x) \quad \Leftrightarrow \quad 200 + 6x^{1/2} = \dfrac{16{,}000}{x} + 200 + 4x^{1/2} \quad \Leftrightarrow \quad 2x^{3/2} = 16{,}000 \quad \Leftrightarrow$ $x = (8{,}000)^{2/3} = 400$ units. To check that this is a minimum, we calculate $c'(x) = \dfrac{-16{,}000}{x^2} + \dfrac{2}{\sqrt{x}} = \dfrac{2}{x^2}(x^{3/2} - 8000)$. This is negative for $x < (8000)^{2/3} = 400$, zero at $x = 400$, and positive for $x > 400$, so c is decreasing on $(0, 400)$ and increasing on $(400, \infty)$. Thus, c has an absolute minimum at $x = 400$. [*Note:* $c''(x)$ is *not* positive for all $x > 0$.]

(iii) The minimum average cost is $c(400) = 40 + 200 + 80 = \$320/\text{unit}$.

66. (a) Show that if the profit $P(x)$ is a maximum, then the marginal revenue equals the marginal cost.

(b) If $C(x) = 16000 + 500x - 1.6x^2 + 0.004x^3$ is the cost function and $p(x) = 1700 - 7x$ is the demand function, find the production level that will maximize profit.

Solution:

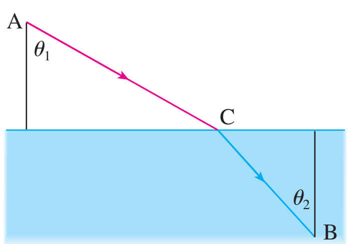
(a) The total profit is $P(x) = R(x) - C(x)$. In order to maximize profit we look for the critical numbers of P, that is, the numbers where the marginal profit is 0. But if $P'(x) = R'(x) - C'(x) = 0$, then $R'(x) = C'(x)$. Therefore, if the profit is a maximum, then the marginal revenue equals the marginal cost.

(b) $C(x) = 16{,}000 + 500x - 1.6x^2 + 0.004x^3$, $p(x) = 1700 - 7x$. Then $R(x) = xp(x) = 1700x - 7x^2$. If the profit is maximum, then $R'(x) = C'(x) \quad \Leftrightarrow \quad 1700 - 14x = 500 - 3.2x + 0.012x^2 \quad \Leftrightarrow \quad 0.012x^2 + 10.8x - 1200 = 0 \quad \Leftrightarrow$ $x^2 + 900x - 100{,}000 = 0 \quad \Leftrightarrow \quad (x + 1000)(x - 100) = 0 \quad \Leftrightarrow \quad x = 100$ (since $x > 0$). The profit is maximized if $P''(x) < 0$, but since $P''(x) = R''(x) - C''(x)$, we can just check the condition $R''(x) < C''(x)$. Now $R''(x) = -14 < -3.2 + 0.024x = C''(x)$ for $x > 0$, so there is a maximum at $x = 100$.

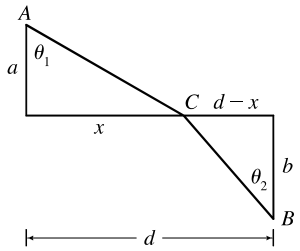
77. Let v_1 be the velocity of light in air and v_2 the velocity of light in water. According to Fermat's Principle, a ray of light will travel from a point A in the air to a point B in the water by a path ACB that minimizes the time taken. Show that

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$$

where θ_1 (the angle of incidence) and θ_2 (the angle of refraction) are as shown. This equation is known as Snell's Law.



Solution:



The total time is

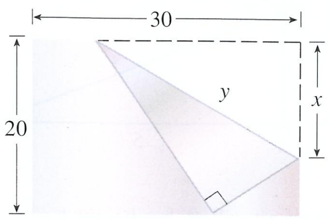
$$T(x) = (\text{time from } A \text{ to } C) + (\text{time from } C \text{ to } B)$$
$$= \frac{\sqrt{a^2+x^2}}{v_1} + \frac{\sqrt{b^2+(d-x)^2}}{v_2}, \quad 0 < x < d$$

$$T'(x) = \frac{x}{v_1\sqrt{a^2+x^2}} - \frac{d-x}{v_2\sqrt{b^2+(d-x)^2}} = \frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2}$$

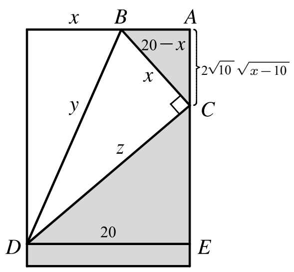
The minimum occurs when $T'(x) = 0 \;\Rightarrow\; \dfrac{\sin\theta_1}{v_1} = \dfrac{\sin\theta_2}{v_2}$.

[*Note:* $T''(x) > 0$]

79. The upper right-hand corner of a piece of paper, 30 inches by 20 inches, as in the figure, is folded over to the bottom edge. How would you fold the paper so as to minimize the length of the fold? In other words, how would you choose x to minimize y?



Solution:



$y^2 = x^2 + z^2$, but triangles CDE and BCA are similar, so

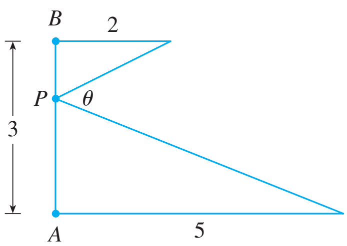
$z/20 = x/2\sqrt{10}\sqrt{x-10} \Rightarrow z = \sqrt{10}x/\sqrt{x-10}$. Thus, we minimize

$f(x) = y^2 = x^2 + 10x^2/(x-10) = x^3/(x-10)$, $10 < x \le 20$.

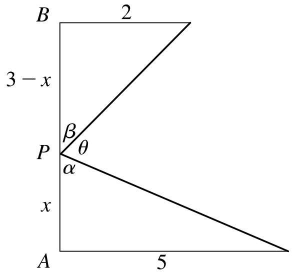
$f'(x) = \frac{(x-10)(3x^2)-x^3}{(x-10)^2} = \frac{x^2[3(x-10)-x]}{(x-10)^2} = \frac{2x^2(x-15)}{(x-10)^2} = 0$ when

$x = 15$. $f'(x) < 0$ when $x < 15$, $f'(x) > 0$ when $x > 15$, so the minimum occurs when $x = 15$ cm.

83. Where should the point P be chosen on the line segment AB so as to maximize the angle θ?



Solution:



From the figure, $\tan\alpha = \dfrac{5}{x}$ and $\tan\beta = \dfrac{2}{3-x}$. Since

$\alpha + \beta + \theta = 180^\circ = \pi$, $\theta = \pi - \tan^{-1}\left(\dfrac{5}{x}\right) - \tan^{-1}\left(\dfrac{2}{3-x}\right) \quad \Rightarrow$

$$\frac{d\theta}{dx} = -\frac{1}{1+\left(\frac{5}{x}\right)^2}\left(-\frac{5}{x^2}\right) - \frac{1}{1+\left(\frac{2}{3-x}\right)^2}\left[\frac{2}{(3-x)^2}\right]$$

$$= \frac{x^2}{x^2+25}\cdot\frac{5}{x^2} - \frac{(3-x)^2}{(3-x)^2+4}\cdot\frac{2}{(3-x)^2}.$$

Now $\dfrac{d\theta}{dx} = 0 \quad \Rightarrow \quad \dfrac{5}{x^2+25} = \dfrac{2}{x^2-6x+13} \quad \Rightarrow \quad 2x^2 + 50 = 5x^2 - 30x + 65 \quad \Rightarrow$

$3x^2 - 30x + 15 = 0 \quad \Rightarrow \quad x^2 - 10x + 5 = 0 \quad \Rightarrow \quad x = 5 \pm 2\sqrt{5}$. We reject the root with the $+$ sign, since it is larger than 3. $d\theta/dx > 0$ for $x < 5 - 2\sqrt{5}$ and $d\theta/dx < 0$ for $x > 5 - 2\sqrt{5}$, so θ is maximized when $|AP| = x = 5 - 2\sqrt{5} \approx 0.53$.