Section 4.5 Summary of Curve Sketching

11. Use the guidelines of this section to sketch the curve.

$$y = \frac{x - x^2}{2 - 3x + x^2}$$

Solution:

$y = f(x) = \dfrac{x - x^2}{2 - 3x + x^2} = \dfrac{x(1-x)}{(1-x)(2-x)} = \dfrac{x}{2-x}$ for $x \neq 1$. There is a hole in the graph at $(1, 1)$.

A. $D = \{x \mid x \neq 1, 2\} = (-\infty, 1) \cup (1, 2) \cup (2, \infty)$ **B.** x-intercept $= 0$, y-intercept $= f(0) = 0$ **C.** No symmetry

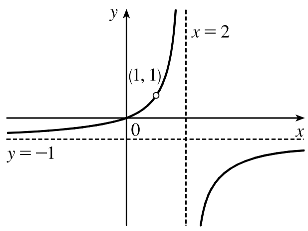
D. $\lim\limits_{x\to\pm\infty} \dfrac{x}{2-x} = -1$, so $y = -1$ is a HA. $\lim\limits_{x\to 2^-} \dfrac{x}{2-x} = \infty$, $\lim\limits_{x\to 2^+} \dfrac{x}{2-x} = -\infty$, so $x = 2$ is a VA.

E. $f'(x) = \dfrac{(2-x)(1) - x(-1)}{(2-x)^2} = \dfrac{2}{(2-x)^2} > 0$ $[x \neq 1, 2]$, so f is increasing on $(-\infty, 1)$, $(1, 2)$, and $(2, \infty)$. **F.** No extrema

G. $f'(x) = 2(2-x)^{-2} \Rightarrow$

$f''(x) = -4(2-x)^{-3}(-1) = \dfrac{4}{(2-x)^3} > 0 \Leftrightarrow x < 2$, so f is CU on $(-\infty, 1)$ and $(1, 2)$, and f is CD on $(2, \infty)$. No IP

H.



37. Use the guidelines of this section to sketch the curve.

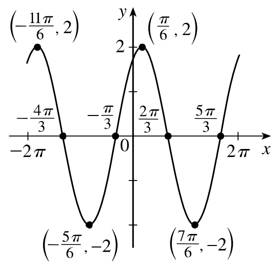
$$y = \sin x + \sqrt{3}\cos x, \quad -2\pi \le x \le 2\pi$$

Solution:

$y = f(x) = \sin x + \sqrt{3}\cos x$, $-2\pi \le x \le 2\pi$ **A.** $D = [-2\pi, 2\pi]$ **B.** y-intercept: $f(0) = \sqrt{3}$; x-intercepts: $f(x) = 0 \Leftrightarrow \sin x = -\sqrt{3}\cos x \Leftrightarrow \tan x = -\sqrt{3} \Leftrightarrow x = -\frac{4\pi}{3}, -\frac{\pi}{3}, \frac{2\pi}{3}$, or $\frac{5\pi}{3}$ **C.** f is periodic with period 2π. **D.** No asymptote **E.** $f'(x) = \cos x - \sqrt{3}\sin x$. $f'(x) = 0 \Leftrightarrow \cos x = \sqrt{3}\sin x \Leftrightarrow \tan x = \dfrac{1}{\sqrt{3}} \Leftrightarrow$ $x = -\frac{11\pi}{6}, -\frac{5\pi}{6}, \frac{\pi}{6}$, or $\frac{7\pi}{6}$. $f'(x) < 0 \Leftrightarrow -\frac{11\pi}{6} < x < -\frac{5\pi}{6}$ or $\frac{\pi}{6} < x < \frac{7\pi}{6}$, so f is decreasing on $\left(-\frac{11\pi}{6}, -\frac{5\pi}{6}\right)$ and $\left(\frac{\pi}{6}, \frac{7\pi}{6}\right)$, and f is increasing on $\left(-2\pi, -\frac{11\pi}{6}\right)$, $\left(-\frac{5\pi}{6}, \frac{\pi}{6}\right)$, and $\left(\frac{7\pi}{6}, 2\pi\right)$. **F.** Local maximum value $f\left(-\frac{11\pi}{6}\right) = f\left(\frac{\pi}{6}\right) = \frac{1}{2} + \sqrt{3}\left(\frac{1}{2}\sqrt{3}\right) = 2$, local minimum value $f\left(-\frac{5\pi}{6}\right) = f\left(\frac{7\pi}{6}\right) = -\frac{1}{2} + \sqrt{3}\left(-\frac{1}{2}\sqrt{3}\right) = -2$

G. $f''(x) = -\sin x - \sqrt{3}\cos x$. $f''(x) = 0 \Leftrightarrow \sin x = -\sqrt{3}\cos x \Leftrightarrow$ $\tan x = -\dfrac{1}{\sqrt{3}} \Leftrightarrow x = -\frac{4\pi}{3}, -\frac{\pi}{3}, \frac{2\pi}{3}$, or $\frac{5\pi}{3}$. $f''(x) > 0 \Leftrightarrow$ $-\frac{4\pi}{3} < x < -\frac{\pi}{3}$ or $\frac{2\pi}{3} < x < \frac{5\pi}{3}$, so f is CU on $\left(-\frac{4\pi}{3}, -\frac{\pi}{3}\right)$ and $\left(\frac{2\pi}{3}, \frac{5\pi}{3}\right)$, and f is CD on $\left(-2\pi, -\frac{4\pi}{3}\right)$, $\left(-\frac{\pi}{3}, \frac{2\pi}{3}\right)$, and $\left(\frac{5\pi}{3}, 2\pi\right)$. There are IPs at $\left(-\frac{4\pi}{3}, 0\right)$, $\left(-\frac{\pi}{3}, 0\right)$, $\left(\frac{2\pi}{3}, 0\right)$, and $\left(\frac{5\pi}{3}, 0\right)$.

H.



54. Use the guidelines of this section to sketch the curve.

$$y = \tan^{-1}\left(\frac{x-1}{x+1}\right)$$

Solution:

$y = f(x) = \tan^{-1}\left(\dfrac{x-1}{x+1}\right)$ **A.** $D = \{x \mid x \neq -1\}$ **B.** x-intercept $= 1$, y-intercept $= f(0) = \tan^{-1}(-1) = -\frac{\pi}{4}$

C. No symmetry **D.** $\displaystyle\lim_{x\to\pm\infty} \tan^{-1}\left(\frac{x-1}{x+1}\right) = \lim_{x\to\pm\infty} \tan^{-1}\left(\frac{1-1/x}{1+1/x}\right) = \tan^{-1} 1 = \frac{\pi}{4}$, so $y = \frac{\pi}{4}$ is a HA.

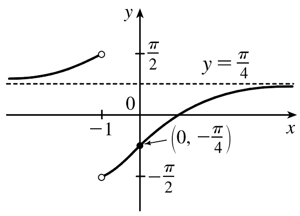
Also $\displaystyle\lim_{x\to-1^+} \tan^{-1}\left(\frac{x-1}{x+1}\right) = -\frac{\pi}{2}$ and $\displaystyle\lim_{x\to-1^-} \tan^{-1}\left(\frac{x-1}{x+1}\right) = \frac{\pi}{2}$.

E. $\displaystyle f'(x) = \frac{1}{1+[(x-1)/(x+1)]^2}\,\frac{(x+1)-(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2+(x-1)^2} = \frac{1}{x^2+1} > 0,$

so f is increasing on $(-\infty,-1)$ and $(-1,\infty)$. **F.** No extreme values **H.**

G. $f''(x) = -2x/(x^2+1)^2 > 0 \quad\Leftrightarrow\quad x < 0$, so f is CU on $(-\infty,-1)$

and $(-1,0)$, and CD on $(0,\infty)$. IP at $\left(0,-\frac{\pi}{4}\right)$



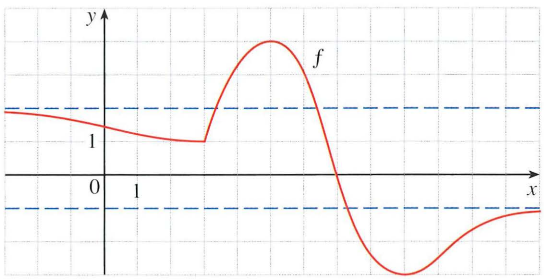
56. The graph of a function f is shown. (The dashed lines indicate horizontal asymptotes.) Find each of the following for the given function $g(x) = \sqrt[3]{f(x)}$.

(a) The domains of g and g'.

(b) The critical numbers of g.

(c) The approximate value of $g'(6)$.

(d) All vertical and horizontal asymptotes of g.



Solution:

$g(x) = \sqrt[3]{f(x)}$

(a) Since the cube-root function is defined for all reals, the domain of g equals the domain of f, $(-\infty,\infty)$.

$\displaystyle g'(x) = \frac{1}{3\left(\sqrt[3]{f(x)}\right)^2} \cdot f'(x) = \frac{f'(x)}{3\left(\sqrt[3]{f(x)}\right)^2}$. Since $f'(3)$ does not exist and $f(7) = 0$, $g'(3)$ and $g'(7)$ do not exist.

The domain of g' is $(-\infty,3) \cup (3,7) \cup (7,\infty)$.

(b) $g'(x) = 0 \;\Rightarrow\; f'(x) = 0 \;\Rightarrow\; x = 5$ or $x = 9$ [there are horizontal tangent lines there]. From part (a), $g'(x)$ does not exist at $x = 3$ and $x = 7$. So the critical numbers of g are 3, 5, 7, and 9.

(c) From part (a), $g'(x) = \dfrac{f'(x)}{3\left(\sqrt[3]{f(x)}\right)^2}$. $g'(6) = \dfrac{f'(6)}{3\left(\sqrt[3]{f(6)}\right)^2} \approx \dfrac{-2}{3\left(\sqrt[3]{3}\right)^2} = -\dfrac{2}{3^{5/3}} \approx -0.32$.

(d) $\lim\limits_{x\to-\infty} g(x) = \lim\limits_{x\to-\infty} \sqrt[3]{f(x)} = \sqrt[3]{2}$ and $\lim\limits_{x\to\infty} g(x) = \lim\limits_{x\to\infty} \sqrt[3]{f(x)} = \sqrt[3]{-1} = -1$, so $y = \sqrt[3]{2}$ and $y = -1$ are horizontal asymptotes. No VA

75. Show that the curve $y = x - \tan^{-1} x$ has two slant asymptotes: $y = x + \frac{\pi}{2}$ and $y = x - \frac{\pi}{2}$. Use this fact to help sketch the curve.

Solution:

$y = f(x) = x - \tan^{-1} x$, $f'(x) = 1 - \dfrac{1}{1+x^2} = \dfrac{1+x^2-1}{1+x^2} = \dfrac{x^2}{1+x^2}$,

$f''(x) = \dfrac{(1+x^2)(2x) - x^2(2x)}{(1+x^2)^2} = \dfrac{2x(1+x^2-x^2)}{(1+x^2)^2} = \dfrac{2x}{(1+x^2)^2}$.

$\lim\limits_{x\to\infty} \left[f(x) - \left(x - \frac{\pi}{2}\right)\right] = \lim\limits_{x\to\infty} \left(\frac{\pi}{2} - \tan^{-1} x\right) = \frac{\pi}{2} - \frac{\pi}{2} = 0$, so $y = x - \frac{\pi}{2}$ is a SA.

Also, $\lim\limits_{x\to-\infty} \left[f(x) - \left(x + \frac{\pi}{2}\right)\right] = \lim\limits_{x\to-\infty} \left(-\frac{\pi}{2} - \tan^{-1} x\right) = -\frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = 0$, so $y = x + \frac{\pi}{2}$ is also a SA. $f'(x) \geq 0$ for all x, with equality $\Leftrightarrow$ $x = 0$, so f is increasing on $\mathbb{R}$. $f''(x)$ has the same sign as x, so f is CD on $(-\infty, 0)$ and CU on $(0, \infty)$. $f(-x) = -f(x)$, so f is an odd function; its graph is symmetric about the origin. f has no local extreme values. Its only IP is at $(0, 0)$.