Section 4.1 Maximum and Minimum Values

45. Find the critical numbers of the function. $f(\theta) = 2\cos\theta + \sin^2\theta$.

Solution:

$f(\theta) = 2\cos\theta + \sin^2\theta \quad \Rightarrow \quad f'(\theta) = -2\sin\theta + 2\sin\theta\,\cos\theta$. $f'(\theta) = 0 \quad \Rightarrow \quad 2\sin\theta\,(\cos\theta - 1) = 0 \quad \Rightarrow \quad \sin\theta = 0$ or $\cos\theta = 1 \quad \Rightarrow \quad \theta = n\pi$ [n an integer] or $\theta = 2n\pi$. The solutions $\theta = n\pi$ include the solutions $\theta = 2n\pi$, so the critical numbers are $\theta = n\pi$.

60. Find the absolute maximum and absolute minimum values of f on the given interval.

$$f(x) = \frac{e^x}{1+x^2}, \quad [0,3]$$

Solution:

$f(x) = \dfrac{e^x}{1+x^2}$, $[0,3]$. $f'(x) = \dfrac{(1+x^2)e^x - e^x(2x)}{(1+x^2)^2} = \dfrac{e^x(x^2-2x+1)}{(1+x^2)^2} = \dfrac{e^x(x-1)^2}{(1+x^2)^2}$. $f'(x) = 0 \quad \Rightarrow$ $(x-1)^2 = 0 \quad \Leftrightarrow \quad x = 1$. $f'(x)$ exists for all real numbers since $1 + x^2$ is never equal to 0. $f(0) = 1$, $f(1) = e/2 \approx 1.359$, and $f(3) = e^3/10 \approx 2.009$. So $f(3) = e^3/10$ is the absolute maximum value and $f(0) = 1$ is the absolute minimum value.

63. Find the absolute maximum and absolute minimum values of f on the given interval.

$$f(x) = x^{-2}\ln x, \quad [\frac{1}{2}, 4]$$

Solution:

$f(x) = x^{-2}\ln x$, $\left[\frac{1}{2}, 4\right]$. $f'(x) = x^{-2}\cdot\dfrac{1}{x} + (\ln x)(-2x^{-3}) = x^{-3} - 2x^{-3}\ln x = x^{-3}(1 - 2\ln x) = \dfrac{1-2\ln x}{x^3}$.

$f'(x) = 0 \quad \Leftrightarrow \quad 1 - 2\ln x = 0 \quad \Leftrightarrow \quad 2\ln x = 1 \quad \Leftrightarrow \quad \ln x = \frac{1}{2} \quad \Leftrightarrow \quad x = e^{1/2} \approx 1.65$. $f'(x)$ does not exist when $x = 0$, which is not in the given interval, $\left[\frac{1}{2}, 4\right]$. $f\left(\frac{1}{2}\right) = \dfrac{\ln 1/2}{(1/2)^2} = \dfrac{\ln 1 - \ln 2}{1/4} = -4\ln 2 \approx -2.773$,

$f\left(e^{1/2}\right) = \dfrac{\ln e^{1/2}}{(e^{1/2})^2} = \dfrac{1/2}{e} = \dfrac{1}{2e} \approx 0.184$, and $f(4) = \dfrac{\ln 4}{4^2} = \dfrac{\ln 4}{16} \approx 0.087$. So $f(e^{1/2}) = \dfrac{1}{2e}$ is the absolute maximum value and $f\left(\frac{1}{2}\right) = -4\ln 2$ is the absolute minimum value.

66. Find the absolute maximum and absolute minimum values of f on the given interval.

$$f(x) = x - 2\tan^{-1}x, \quad [0,4]$$

Solution:

$f(x) = x - 2\tan^{-1}x$, $[0,4]$. $f'(x) = 1 - 2\cdot\dfrac{1}{1+x^2} = 0 \quad \Leftrightarrow \quad 1 = \dfrac{2}{1+x^2} \quad \Leftrightarrow \quad 1 + x^2 = 2 \quad \Leftrightarrow \quad x^2 = 1 \quad \Leftrightarrow$ $x = \pm 1$. $f(0) = 0$, $f(1) = 1 - \frac{\pi}{2} \approx -0.57$, and $f(4) = 4 - 2\tan^{-1}4 \approx 1.35$. So $f(4) = 4 - 2\tan^{-1}4$ is the absolute maximum value and $f(1) = 1 - \frac{\pi}{2}$ is the absolute minimum value.