Section 3.4 The Chain Rule

46. Find the derivative of the function. $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$.

Solution:

$$y = \sqrt{x + \sqrt{x + \sqrt{x}}} \quad \Rightarrow \quad y' = \frac{1}{2}\left(x + \sqrt{x + \sqrt{x}}\right)^{-1/2}\left[1 + \frac{1}{2}\left(x + \sqrt{x}\right)^{-1/2}\left(1 + \frac{1}{2}x^{-1/2}\right)\right]$$

48. Find the derivative of the function. $y = 2^{3^{4^x}}$.

Solution:

$$y = 2^{3^{4^x}} \quad \Rightarrow$$

$$y' = 2^{3^{4^x}}(\ln 2)\frac{d}{dx}3^{4^x} = 2^{3^{4^x}}(\ln 2)\,3^{4^x}(\ln 3)\frac{d}{dx}4^x = 2^{3^{4^x}}(\ln 2)\,3^{4^x}(\ln 3)\,4^x(\ln 4) = (\ln 2)(\ln 3)(\ln 4)4^x 3^{4^x} 2^{3^{4^x}}$$

69. A table of values for f, g, f' and g' is given.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	4	6
2	1	8	5	7
3	7	2	7	9

(a) If $h(x) = f(g(x))$, find $h'(1)$.

(b) If $H(x) = g(f(x))$, find $H'(1)$.

Solution:

(a) $h(x) = f(g(x)) \quad \Rightarrow \quad h'(x) = f'(g(x)) \cdot g'(x)$, so $h'(1) = f'(g(1)) \cdot g'(1) = f'(2) \cdot 6 = 5 \cdot 6 = 30$.

(b) $H(x) = g(f(x)) \quad \Rightarrow \quad H'(x) = g'(f(x)) \cdot f'(x)$, so $H'(1) = g'(f(1)) \cdot f'(1) = g'(3) \cdot 4 = 9 \cdot 4 = 36$.

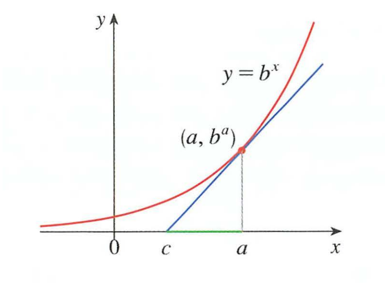
78. If g is a twice differentiable function and $f(x) = xg(x^2)$, find f'' in terms of g, g', and g''.

Solution:

$$f(x) = xg(x^2) \quad \Rightarrow \quad f'(x) = xg'(x^2)\,2x + g(x^2) \cdot 1 = 2x^2 g'(x^2) + g(x^2) \quad \Rightarrow$$

$$f''(x) = 2x^2 g''(x^2)\,2x + g'(x^2)\,4x + g'(x^2)\,2x = 4x^3 g''(x^2) + 4xg'(x^2) + 2xg'(x^2) = 6xg'(x^2) + 4x^3 g''(x^2)$$

99. Let c be the x-intercept of the tangent line to the curve $y = b^x (b > 0,\ b \neq 1)$ at the point (a, b^a). Show that the distance between the point $(a, 0)$ and $(c, 0)$ is the same for all values of a.



Solution:

$y = b^x \quad \Rightarrow \quad y' = b^x \ln b$, so the slope of the tangent line to the curve $y = b^x$ at the point (a, b^a) is $b^a \ln b$. An equation of this tangent line is then $y - b^a = b^a \ln b\,(x - a)$. If c is the x-intercept of this tangent line, then $0 - b^a = b^a \ln b\,(c - a) \quad \Rightarrow$ $-1 = \ln b\,(c - a) \quad \Rightarrow \quad \dfrac{-1}{\ln b} = c - a \quad \Rightarrow \quad |c - a| = \left|\dfrac{-1}{\ln b}\right| = \dfrac{1}{|\ln b|}$. The distance between $(a, 0)$ and $(c, 0)$ is $|c - a|$, and this distance is the constant $\dfrac{1}{|\ln b|}$ for any a. [*Note*: The absolute value is needed for the case $0 < b < 1$ because $\ln b$ is negative there. If $b > 1$, we can write $a - c = 1/(\ln b)$ as the constant distance between $(a, 0)$ and $(c, 0)$.]