Section 3.2 The Product and Quotient Rules

44. If $g(x) = \frac{x}{e^x}$, find $g^{(n)}(x)$.

Solution:

$$g(x) = \frac{x}{e^x} \quad \Rightarrow \quad g'(x) = \frac{e^x \cdot 1 - x \cdot e^x}{(e^x)^2} = \frac{e^x(1-x)}{(e^x)^2} = \frac{1-x}{e^x} \quad \Rightarrow$$

$$g''(x) = \frac{e^x \cdot (-1) - (1-x)e^x}{(e^x)^2} = \frac{e^x[-1-(1-x)]}{(e^x)^2} = \frac{x-2}{e^x} \quad \Rightarrow$$

$$g'''(x) = \frac{e^x \cdot 1 - (x-2)e^x}{(e^x)^2} = \frac{e^x[1-(x-2)]}{(e^x)^2} = \frac{3-x}{e^x} \quad \Rightarrow$$

$$g^{(4)}(x) = \frac{e^x \cdot (-1) - (3-x)e^x}{(e^x)^2} = \frac{e^x[-1-(3-x)]}{(e^x)^2} = \frac{x-4}{e^x}.$$

The pattern suggests that $g^{(n)}(x) = \dfrac{(x-n)(-1)^n}{e^x}$. (We could use mathematical induction to prove this formula.)

50. If $f(2) = 10$ and $f'(x) = x^2 f(x)$ for all x, find $f''(2)$.

Solution:

$f'(x) = x^2 f(x) \quad \Rightarrow \quad f''(x) = x^2 f'(x) + f(x) \cdot 2x$. Now $f'(2) = 2^2 f(2) = 4(10) = 40$, so

$f''(2) = 2^2(40) + 10(4) = 200$.

54. If f is a differentiable function, find an expression for the derivative of each of the following functions.
(a) $y = x^2 f(x)$ (b) $y = \frac{f(x)}{x^2}$ (c) $y = \frac{x^2}{f(x)}$ (d) $y = \frac{1+xf(x)}{\sqrt{x}}$

Solution:

(a) $y = x^2 f(x) \quad \Rightarrow \quad y' = x^2 f'(x) + f(x)(2x)$

(b) $y = \dfrac{f(x)}{x^2} \quad \Rightarrow \quad y' = \dfrac{x^2 f'(x) - f(x)(2x)}{(x^2)^2} = \dfrac{xf'(x) - 2f(x)}{x^3}$

(c) $y = \dfrac{x^2}{f(x)} \quad \Rightarrow \quad y' = \dfrac{f(x)(2x) - x^2 f'(x)}{[f(x)]^2}$

(d) $y = \dfrac{1+xf(x)}{\sqrt{x}} \quad \Rightarrow$

$$y' = \frac{\sqrt{x}\,[xf'(x) + f(x)] - [1 + xf(x)]\,\dfrac{1}{2\sqrt{x}}}{(\sqrt{x})^2}$$

$$= \frac{x^{3/2} f'(x) + x^{1/2} f(x) - \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{1/2} f(x)}{x} \cdot \frac{2x^{1/2}}{2x^{1/2}} = \frac{xf(x) + 2x^2 f'(x) - 1}{2x^{3/2}}$$

58. Use the method of Hint to compute $Q'(0)$, where

$$Q(x) = \frac{1 + x + x^2 + xe^x}{1 - x + x^2 - xe^x}$$

Hint: Instead of finding $Q'(x)$ first, let $f(x)$ be the numerator and $g(x)$ the denominator of $Q(x)$ and compute $Q'(x)$ from $f(0)$, $f'(0)$, $g(0)$, and $g'(0)$.

Solution:

$Q = \dfrac{f}{g} \quad \Rightarrow \quad Q' = \dfrac{gf' - fg'}{g^2}$. For $f(x) = 1 + x + x^2 + xe^x$, $f'(x) = 1 + 2x + xe^x + e^x$,

and for $g(x) = 1 - x + x^2 - xe^x$, $g'(x) = -1 + 2x - xe^x - e^x$.

Thus, $Q'(0) = \dfrac{g(0)f'(0) - f(0)g'(0)}{[g(0)]^2} = \dfrac{1 \cdot 2 - 1 \cdot (-2)}{1^2} = \dfrac{4}{1} = 4$.