Section 2.7 Derivatives and Rates of Change

34. If the tangent line to $y = f(x)$ at $(4, 3)$ passes through the point $(0, 2)$, find $f(4)$ and $f'(4)$.

Solution:

Since $(4, 3)$ is on $y = f(x)$, $f(4) = 3$. The slope of the tangent line between $(0, 2)$ and $(4, 3)$ is $\frac{1}{4}$, so $f'(4) = \frac{1}{4}$.

36. A particle moves along a straight line with equation of motion $s = f(t)$, where s is measured in meters and t in seconds. Find the velocity and the speed when $t = 4$.

$$f(t) = 10 + \frac{45}{t+1}$$

Solution:

$$v(4) = f'(4) = \lim_{h\to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h\to 0} \frac{\left(10 + \dfrac{45}{4+h+1}\right) - \left(10 + \dfrac{45}{4+1}\right)}{h} = \lim_{h\to 0} \frac{\dfrac{45}{5+h} - 9}{h}$$

$$= \lim_{h\to 0} \frac{45 - 9(5+h)}{h(5+h)} = \lim_{h\to 0} \frac{-9h}{h(5+h)} = \lim_{h\to 0} \frac{-9}{5+h} = -\frac{9}{5} \text{ m/s.}$$

The speed when $t = 4$ is $\left|-\frac{9}{5}\right| = \frac{9}{5}$ m/s.

58. Determine whether $f'(0)$ exists.

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Solution:

Since $f(x) = x^2 \sin(1/x)$ when $x \neq 0$ and $f(0) = 0$, we have

$f'(0) = \lim\limits_{h\to 0} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h\to 0} \dfrac{h^2 \sin(1/h) - 0}{h} = \lim\limits_{h\to 0} h \sin(1/h)$. Since $-1 \leq \sin \dfrac{1}{h} \leq 1$, we have

$-|h| \leq |h| \sin \dfrac{1}{h} \leq |h| \quad \Rightarrow \quad -|h| \leq h \sin \dfrac{1}{h} \leq |h|$. Because $\lim\limits_{h\to 0} (-|h|) = 0$ and $\lim\limits_{h\to 0} |h| = 0$, we know that

$\lim\limits_{h\to 0} \left(h \sin \dfrac{1}{h}\right) = 0$ by the Squeeze Theorem. Thus, $f'(0) = 0$.