Section 16.7 Surface Integrals

14. Evaluate the surface integral. $\iint_S y^2z^2 dS$, S is the part of the cone $y = \sqrt{x^2+z^2}$ given by $0 \le y \le 5$.

Solution:

Using x and z as parameters, we have $\mathbf{r}(x,z) = x\,\mathbf{i} + \sqrt{x^2+z^2}\,\mathbf{j} + z\,\mathbf{k}$, $x^2+z^2 \le 25$. Then

$\mathbf{r}_x \times \mathbf{r}_z = \left(\mathbf{i} + \frac{x}{\sqrt{x^2+z^2}}\mathbf{j}\right) \times \left(\frac{z}{\sqrt{x^2+z^2}}\mathbf{j} + \mathbf{k}\right) = \frac{x}{\sqrt{x^2+z^2}}\mathbf{i} - \mathbf{j} + \frac{z}{\sqrt{x^2+z^2}}\mathbf{k}$ and

$|\mathbf{r}_x \times \mathbf{r}_z| = \sqrt{\frac{x^2}{x^2+z^2} + 1 + \frac{z^2}{x^2+z^2}} = \sqrt{\frac{x^2+z^2}{x^2+z^2} + 1} = \sqrt{2}$. Thus

$$\begin{aligned}\iint_S y^2z^2\,dS &= \iint_{x^2+z^2\le 25} (x^2+z^2)z^2\sqrt{2}\,dA = \sqrt{2}\int_0^{2\pi}\int_0^5 r^2(r\sin\theta)^2\,r\,dr\,d\theta \\ &= \sqrt{2}\int_0^{2\pi}\sin^2\theta\,d\theta\int_0^5 r^5\,dr = \sqrt{2}\left[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi}\left[\tfrac{1}{6}r^6\right]_0^5 \\ &= \sqrt{2}\,(\pi)\cdot\tfrac{1}{6}(15{,}625 - 0) = \frac{15{,}625\sqrt{2}}{6}\pi\end{aligned}$$

30. Evaluate the surface integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$ for the given vector field $\mathbf{F}$ and the oriented surface S. In other words, find the flux of $\mathbf{F}$ across S. For closed surfaces, use the positive (outward) orientation.
$\mathbf{F}(\mathrm{x,y,z}) = \mathrm{x}\,\mathbf{i} + \mathrm{y}\,\mathbf{j} + 5\,\mathbf{k}$, S is the boundary of the region enclosed by the cylinder $x^2+z^2=1$ and the planes $y=0$ and $x+y=2$.

Solution:

$\mathbf{F}(x,y,z) = x\,\mathbf{i} + y\,\mathbf{j} + 5\,\mathbf{k}$. Here S consists of three surfaces: S_1, the lateral surface of the cylinder $x^2+z^2=1$; S_2, the front formed by the plane $x+y=2$; and the back, S_3, in the plane $y=0$.

On S_1: $\mathbf{r}(\theta,y) = \sin\theta\,\mathbf{i} + y\,\mathbf{j} + \cos\theta\,\mathbf{k}$. $\mathbf{F}(\mathbf{r}(\theta,y)) = \sin\theta\,\mathbf{i} + y\,\mathbf{j} + 5\,\mathbf{k}$ and $\mathbf{r}_\theta \times \mathbf{r}_y = \sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{k}$ $\Rightarrow$

$$\begin{aligned}\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} &= \int_0^{2\pi}\int_0^{2-\sin\theta}(\sin^2\theta + 5\cos\theta)\,dy\,d\theta \\ &= \int_0^{2\pi}(2\sin^2\theta + 10\cos\theta - \sin^3\theta - 5\sin\theta\cos\theta)\,d\theta = 2\pi\end{aligned}$$

On S_2: $\mathbf{r}(x,z) = x\,\mathbf{i} + (2-x)\,\mathbf{j} + z\,\mathbf{k}$. $\mathbf{F}(\mathbf{r}(x,z)) = x\,\mathbf{i} + (2-x)\,\mathbf{j} + 5\,\mathbf{k}$ and $\mathbf{r}_z \times \mathbf{r}_x = \mathbf{i} + \mathbf{j}$.

$$\iint_{S_2}\mathbf{F}\cdot d\mathbf{S} = \iint_{x^2+z^2\le 1}[x + (2-x)]\,dA = 2\pi$$

On S_3: $\mathbf{F}(\mathbf{r}(x,z)) = x\,\mathbf{i} + 5\,\mathbf{k}$. The surface is oriented in the negative y-direction so that $\mathbf{n} = -\mathbf{j}$ and by (8), $\iint_{S_3}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_3}\mathbf{F}\cdot\mathbf{n}\,d\mathbf{S} = 0$. Hence, $\iint_S \mathbf{F}\cdot d\mathbf{S} = 4\pi$.

44. Seawater has density 1025 kg/m^3 and flows in a velocity field $\mathbf{v} = \mathrm{y}\,\mathbf{i} + \mathrm{x}\,\mathbf{j}$, where x, y, and z are measured in meters and the components of $\mathbf{v}$ in meters per second. Find the rate of flow outward through the hemisphere $x^2+y^2+z^2=9$, $z \ge 0$.

Solution:

A parametric representation for the hemisphere S is $\mathbf{r}(\phi, \theta) = 3\sin\phi\,\cos\theta\,\mathbf{i} + 3\sin\phi\,\sin\theta\,\mathbf{j} + 3\cos\phi\,\mathbf{k}$, $0 \le \phi \le \pi/2$, $0 \le \theta \le 2\pi$. Then $\mathbf{r}_\phi = 3\cos\phi\,\cos\theta\,\mathbf{i} + 3\cos\phi\,\sin\theta\,\mathbf{j} - 3\sin\phi\,\mathbf{k}$, $\mathbf{r}_\theta = -3\sin\phi\sin\theta\,\mathbf{i} + 3\sin\phi\cos\theta\,\mathbf{j}$, and the outward orientation is given by $\mathbf{r}_\phi \times \mathbf{r}_\theta = 9\sin^2\phi\,\cos\theta\,\mathbf{i} + 9\sin^2\phi\,\sin\theta\,\mathbf{j} + 9\sin\phi\,\cos\phi\,\mathbf{k}$. The rate of flow through S is

$$\iint_S \rho\mathbf{v}\cdot\,d\mathbf{S} = \rho\int_0^{\pi/2}\int_0^{2\pi}(3\sin\phi\,\sin\theta\,\mathbf{i} + 3\sin\phi\,\cos\theta\,\mathbf{j})\cdot\left(9\sin^2\phi\,\cos\theta\,\mathbf{i} + 9\sin^2\phi\sin\theta\,\mathbf{j} + 9\sin\phi\,\cos\phi\,\mathbf{k}\right)d\theta\,d\phi$$

$$= 27\rho\int_0^{\pi/2}\int_0^{2\pi}\left(\sin^3\phi\,\sin\theta\,\cos\theta + \sin^3\phi\,\sin\theta\,\cos\theta\right)d\theta\,d\phi = 54\rho\int_0^{\pi/2}\sin^3\phi\,d\phi\int_0^{2\pi}\sin\theta\,\cos\theta\,d\theta$$

$$= 54\rho\left[-\tfrac{1}{3}(2+\sin^2\phi)\cos\phi\right]_0^{\pi/2}\left[\tfrac{1}{2}\sin^2\theta\right]_0^{2\pi} = 0\text{ kg/s}$$