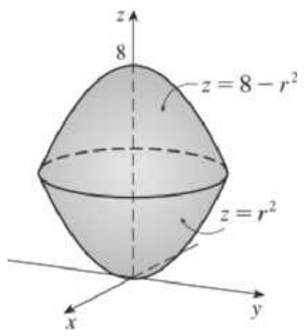
Section 15.7 Triple Integrals in Cylindrical Coordinates

11. Sketch the solid described by the given inequalities. $r^2 \leq z \leq 8 - r^2$

Solution:

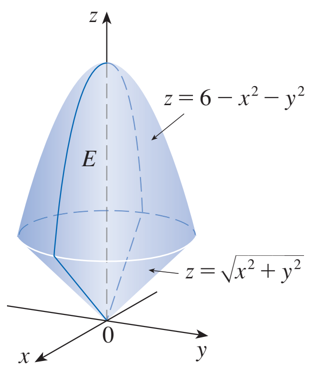


$z = r^2 \quad \Leftrightarrow \quad z = x^2 + y^2$, a circular paraboloid opening upward with vertex the origin, and $z = 8 - r^2 \quad \Leftrightarrow \quad z = 8 - (x^2 + y^2)$, a circular paraboloid opening downward with vertex $(0, 0, 8)$. The paraboloids intersect when $r^2 = 8 - r^2 \quad \Leftrightarrow \quad r^2 = 4$. Thus $r^2 \leq z \leq 8 - r^2$ describes the solid above the paraboloid $z = x^2 + y^2$ and below the paraboloid $z = 8 - x^2 - y^2$ for $x^2 + y^2 \leq 4$.

16. (a) Express the triple integral $\iiint_E f(x, y, z)\, dV$ as an iterated integral in cylindrical coordinates for the given function f and solid region E.

(b) Evaluate the iterated integral.

$f(x, y, z) = xy$



Solution:

(a) In cylindrical coordinates, the region E is bounded above by the paraboloid $z = 6 - r^2$ and below by the cone $z = r$. The paraboloid and cone intersect when $6 - r^2 = r \quad \Rightarrow \quad r^2 + r - 6 = 0 \quad \Rightarrow \quad r = 2\ (r > 0)$, so the region can be described as $E = \{(r, \theta, z) \mid 0 \leq \theta \leq 2\pi, 0 \leq r \leq 2, r \leq z \leq 6 - r^2\}$. Then

$\iiint_E (xy)\, dV = \int_0^{2\pi} \int_0^2 \int_r^{6-r^2} r\cos\theta \cdot r\sin\theta \cdot r\, dz\, dr\, d\theta.$

(b)
$$\int_0^{2\pi} \int_0^2 \int_r^{6-r^2} r^3 \cos\theta \sin\theta\, dz\, dr\, d\theta = \int_0^{2\pi} \int_0^2 r^3 \cos\theta \sin\theta \Big[z \Big]_{z=r}^{z=6-r^2} dr\, d\theta$$
$$= \int_0^{2\pi} \int_0^2 r^3 \cos\theta \sin\theta\, (6r^3 - r^4 - r^5)\, dr\, d\theta$$
$$= \int_0^{2\pi} \cos\theta \sin\theta\, d\theta \int_0^2 (6r^3 - r^4 - r^5)\, dr = 0 \cdot \int_0^2 (6r^3 - r^4 - r^5)\, dr = 0$$

22. Evaluate $\iiint_E (x-y)dV$, where E is the solid that lies between the cylinders $x^2+y^2=1$ and $x^2+y^2=16$, above the xy-plane, and below the plane $z=y+4$.

Solution:

In cylindrical coordinates E is bounded by the planes $z=0$, $z=r\sin\theta+4$ and the cylinders $r=1$ and $r=4$, so E is given by $\{(r,\theta,z)\mid 0\le\theta\le 2\pi,\ 1\le r\le 4,\ 0\le z\le r\sin\theta+4\}$. Thus

$$\begin{aligned}
\iiint_E (x-y)\,dV &= \int_0^{2\pi}\int_1^4\int_0^{r\sin\theta+4}(r\cos\theta-r\sin\theta)\,r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_1^4 (r^2\cos\theta-r^2\sin\theta)[\,z\,]_{z=0}^{z=r\sin\theta+4}\,dr\,d\theta \\
&= \int_0^{2\pi}\int_1^4 (r^2\cos\theta-r^2\sin\theta)(r\sin\theta+4)\,dr\,d\theta \\
&= \int_0^{2\pi}\int_1^4 \left[r^3(\sin\theta\cos\theta-\sin^2\theta)+4r^2(\cos\theta-\sin\theta)\right]dr\,d\theta \\
&= \int_0^{2\pi}\left[\tfrac14 r^4(\sin\theta\cos\theta-\sin^2\theta)+\tfrac43 r^3(\cos\theta-\sin\theta)\right]_{r=1}^{r=4}d\theta \\
&= \int_0^{2\pi}\left[\left(64-\tfrac14\right)(\sin\theta\cos\theta-\sin^2\theta)+\left(\tfrac{256}{3}-\tfrac43\right)(\cos\theta-\sin\theta)\right]d\theta \\
&= \int_0^{2\pi}\left[\tfrac{255}{4}(\sin\theta\cos\theta-\sin^2\theta)+84(\cos\theta-\sin\theta)\right]d\theta \\
&= \left[\tfrac{255}{4}\left(\tfrac12\sin^2\theta-\left(\tfrac12\theta-\tfrac14\sin 2\theta\right)\right)+84\left(\sin\theta+\cos\theta\right)\right]_0^{2\pi} = \tfrac{255}{4}(-\pi)+84(1)-0-84(1) = -\tfrac{255}{4}\pi
\end{aligned}$$

30. Find the mass of a ball B given by $x^2+y^2+z^2\le a^2$ if the density at any point is proportional to its distance from the z-axis.

Solution:

Since density is proportional to the distance from the z-axis, we can say $\rho(x,y,z)=K\sqrt{x^2+y^2}$. Then

$$\begin{aligned}
m &= 2\int_0^{2\pi}\int_0^a\int_0^{\sqrt{a^2-r^2}} Kr^2\,dz\,dr\,d\theta = 2K\int_0^{2\pi}\int_0^a r^2\sqrt{a^2-r^2}\,dr\,d\theta \\
&= 2K\int_0^{2\pi}\left[\tfrac18 r(2r^2-a^2)\sqrt{a^2-r^2}+\tfrac18 a^4\sin^{-1}(r/a)\right]_{r=0}^{r=a}d\theta = 2K\int_0^{2\pi}\left[\left(\tfrac18 a^4\right)\left(\tfrac{\pi}{2}\right)\right]d\theta = \tfrac14 a^4\pi^2 K
\end{aligned}$$