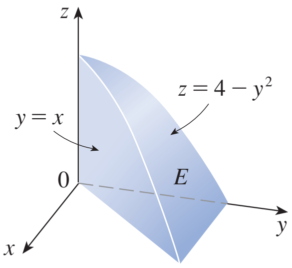
Section 15.6 Triple Integrals

10. (a) Express the triple integral $\iiint_E f(x, y, z)\, dV$ as an iterated integral for the given function f and solid region E.
(b) Evaluate the iterated integral.



Solution:

(a) The solid region E can be described as $E = \{(x, y, z) \mid 0 \le x \le y, 0 \le y \le 2, 0 \le z \le 4 - y^2\}$.

Thus, $\iiint_E xy\, dV = \int_0^2 \int_0^y \int_0^{4-y^2} xy\, dz\, dx\, dy$.

(b) $\int_0^2 \int_0^y \int_0^{4-y^2} xy\, dz\, dx\, dy = \int_0^2 \int_0^y xy\, [z]_{z=0}^{z=4-y^2}\, dx\, dy = \int_0^2 \int_0^y xy(4 - y^2)\, dx\, dy = \int_0^2 \int_0^y x(4y - y^3)\, dx\, dy$

$= \int_0^2 (4y - y^3) \left[\frac{x^2}{2}\right]_{x=0}^{x=y} dy = \frac{1}{2} \int_0^2 (4y^3 - y^5)\, dy = \frac{1}{2} \left[y^4 - \frac{y^6}{6}\right]_0^2 = \frac{8}{3}$

26. Use a triple integral to find the volume of the given solid. The solid enclosed by the cylinder $x^2 + z^2 = 4$ and the planes $y = -1$ and $y + z = 4$.

Solution:

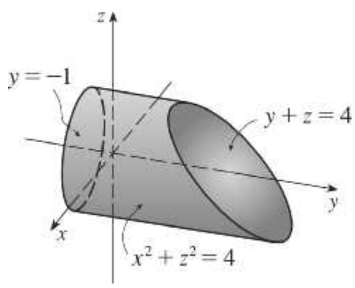
Here $E = \left\{(x, y, z) \mid -1 \le y \le 4 - z, x^2 + z^2 \le 4\right\}$, so

$$V = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{-1}^{4-z} dy\, dz\, dx = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - z + 1)\, dz\, dx$$

$$= \int_{-2}^{2} \left[5z - \tfrac{1}{2} z^2\right]_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} dx = \int_{-2}^{2} 10\sqrt{4 - x^2}\, dx$$

$$= 10\left[\tfrac{x}{2}\sqrt{4 - x^2} + 2\sin^{-1}\left(\tfrac{x}{2}\right)\right]_{-2}^{2}$$ [using trigonometric substitution or Formula 30 in the Table of Integrals]

$$= 10\left[2\sin^{-1}(1) - 2\sin^{-1}(-1)\right] = 20\left(\tfrac{\pi}{2} - \left(-\tfrac{\pi}{2}\right)\right) = 20\pi$$



Alternatively, use polar coordinates to evaluate the double integral:

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (5 - z)\, dz\, dx = \int_0^{2\pi} \int_0^2 (5 - r\sin\theta)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\tfrac{5}{2} r^2 - \tfrac{1}{3} r^3 \sin\theta\right]_{r=0}^{r=2} d\theta$$

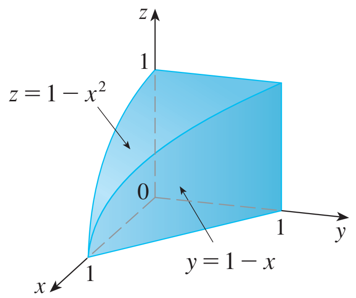
$$= \int_0^{2\pi} \left(10 - \tfrac{8}{3}\sin\theta\right) d\theta$$

$$= 10\theta + \tfrac{8}{3}\cos\theta\Big]_0^{2\pi} = 20\pi$$

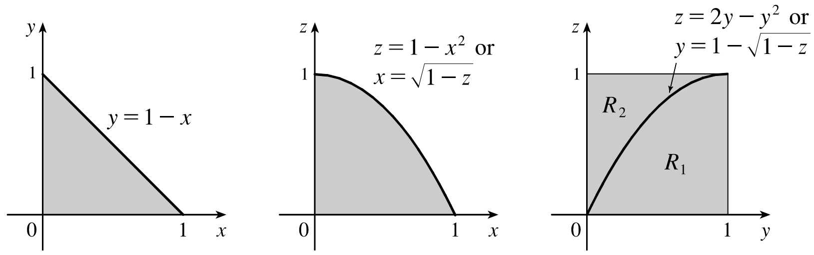
38. The figure shows the region of integration for the integral

$$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x, y, z) dy dz dx$$

Rewrite this integral as an equivalent iterated integral in the five other orders.



Solution:



The projections of E onto the xy- and xz-planes are as in the first two diagrams and so

$$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x,y,z)\,dy\,dz\,dx = \int_0^1 \int_0^{\sqrt{1-z}} \int_0^{1-x} f(x,y,z)\,dy\,dx\,dz$$

$$= \int_0^1 \int_0^{1-y} \int_0^{1-x^2} f(x,y,z)\,dz\,dx\,dy = \int_0^1 \int_0^{1-x} \int_0^{1-x^2} f(x,y,z)\,dz\,dy\,dx$$

Now the surface $z = 1 - x^2$ intersects the plane $y = 1 - x$ in a curve whose projection in the yz-plane is $z = 1 - (1-y)^2$ or $z = 2y - y^2$. So we must split up the projection of E on the yz-plane into two regions as in the third diagram. For (y,z) in R_1, $0 \le x \le 1 - y$ and for (y,z) in R_2, $0 \le x \le \sqrt{1-z}$, and so the given integral is also equal to

$$\int_0^1 \int_0^{1-\sqrt{1-z}} \int_0^{\sqrt{1-z}} f(x,y,z)\,dx\,dy\,dz + \int_0^1 \int_{1-\sqrt{1-z}}^1 \int_0^{1-y} f(x,y,z)\,dx\,dy\,dz$$

$$= \int_0^1 \int_0^{2y-y^2} \int_0^{1-y} f(x,y,z)\,dx\,dz\,dy + \int_0^1 \int_{2y-y^2}^1 \int_0^{\sqrt{1-z}} f(x,y,z)\,dx\,dz\,dy.$$

42. Evaluate the triple integral using only geometric interpretation and symmetry.

$$\iiint_B (z^3 + \sin y + 3)dV, \text{ where } B \text{ is the unit ball } x^2 + y^2 + z^2 \le 1.$$

Solution:

We can write $\iiint_B (z^3 + \sin y + 3)\,dV = \iiint_B z^3\,dV + \iiint_B \sin y\,dV + \iiint_B 3\,dV$. But z^3 is an odd function with respect to z and the region B is symmetric about the xy-plane, so $\iiint_B z^3\,dV = 0$. Similarly, $\sin y$ is an odd function with respect to y and B is symmetric about the xz-plane, so $\iiint_B \sin y\,dV = 0$. Thus $\iiint_B (z^3 + \sin y + 3)\,dV = \iiint_B 3\,dV = 3 \cdot V(B) = 3 \cdot \frac{4}{3}\pi(1)^3 = 4\pi$.

58. **Average Value** The *average value* of a function $f(x,y,z)$ over a solid region E is defined to be

$$f_{avg} = \frac{1}{V(E)} \iiint_E f(x,y,z)dV$$

where $V(E)$ is the volume of E. For instance, if ρ is a density function, then ρ_{ave} is the average density of E. Find the average height of the points in the solid hemisphere $x^2 + y^2 + z^2 \le 1$, $z \ge 0$.

Solution:

The height of each point is given by its z-coordinate, so the average height of the points in

$E = \{(x, y, z) \mid x^2 + y^2 + z^2 \le 1,\ z \ge 0\}$ is

$$\frac{1}{V(E)} \iiint_E z\,dV$$

Here $V(E) = \frac{1}{2} \cdot \frac{4}{3}\pi(1)^3 = \frac{2}{3}\pi$ [half the volume of a sphere], so

$$\frac{1}{V(E)} \iiint_E z\,dV = \frac{1}{2\pi/3} \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} z\,dz\,dy\,dx = \frac{3}{2\pi} \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left[\tfrac{1}{2}z^2\right]_{z=0}^{z=\sqrt{1-x^2-y^2}} dy\,dx$$

$$= \frac{3}{2\pi} \cdot \frac{1}{2} \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2)\,dy\,dx = \frac{3}{4\pi} \int_0^{2\pi} \int_0^1 (1 - r^2)\,r\,dr\,d\theta$$

$$= \frac{3}{4\pi} \int_0^{2\pi} d\theta \int_0^1 (r - r^3)\,dr = \frac{3}{4\pi}(2\pi)\left[\tfrac{1}{2}r^2 - \tfrac{1}{4}r^4\right]_0^1 = \frac{3}{2}\left(\frac{1}{4}\right) = \frac{3}{8}$$