# Section 15.3 Double Integrals in Polar Coordinates

13. Evaluate the given integral by changing to polar coordinates.

$\iint_D e^{-x^2-y^2}\,dA$, where $D$ is the region bounded by the semicircle $x = \sqrt{4-y^2}$ and the $y$-axis.

**Solution:**

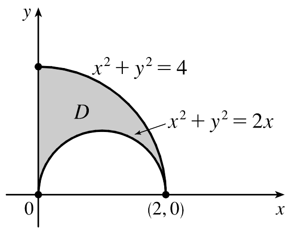
$$\iint_D e^{-x^2-y^2}\,dA = \int_{-\pi/2}^{\pi/2}\int_0^2 e^{-r^2}\,r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} d\theta \int_0^2 re^{-r^2}\,dr$$

$$= \left[\,\theta\,\right]_{-\pi/2}^{\pi/2}\left[-\tfrac{1}{2}e^{-r^2}\right]_0^2 = \pi\left(-\tfrac{1}{2}\right)(e^{-4}-e^0) = \tfrac{\pi}{2}(1-e^{-4})$$

16. Evaluate the given integral by changing to polar coordinates.

$\iint_D x\,dA$, where $D$ is the region in the first quadrant that lies between the circles $x^2+y^2=4$ and $x^2+y^2=2x$.

**Solution:**




$$\iint_D x\,dA = \iint\limits_{\substack{x^2+y^2\le 4\\ x\ge 0,\, y\ge 0}} x\,dA \;-\; \iint\limits_{\substack{(x-1)^2+y^2\le 1\\ y\ge 0}} x\,dA$$

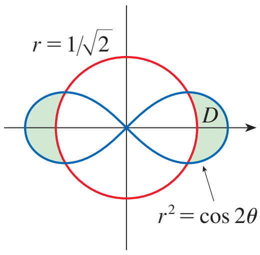
$$= \int_0^{\pi/2}\int_0^2 r^2\cos\theta\,dr\,d\theta - \int_0^{\pi/2}\int_0^{2\cos\theta} r^2\cos\theta\,dr\,d\theta$$

$$= \int_0^{\pi/2}\tfrac{1}{3}(8\cos\theta)\,d\theta - \int_0^{\pi/2}\tfrac{1}{3}(8\cos^4\theta)\,d\theta$$

$$= \tfrac{8}{3}[\sin\theta]_0^{\pi/2} - \tfrac{8}{12}\left[\cos^3\theta\sin\theta + \tfrac{3}{2}(\theta+\sin\theta\cos\theta)\right]_0^{\pi/2}$$

$$= \tfrac{8}{3} - \tfrac{2}{3}\left[0+\tfrac{3}{2}\left(\tfrac{\pi}{2}\right)\right] = \tfrac{16-3\pi}{6}$$

20. Use a double integral to find the area of the region $D$.




**Solution:**

By symmetry, the area of the region is 4 times the area of the region $D$ in the first quadrant between the circle $r = 1/\sqrt{2}$ and the curve $r^2 = \cos 2\theta \;\Rightarrow\; r = \sqrt{\cos 2\theta}$. The curves intersect in the first quadrant when $\cos 2\theta = \left(\frac{1}{\sqrt{2}}\right)^2 \;\Rightarrow\; \cos 2\theta = \frac{1}{2} \;\Rightarrow\; 2\theta = \frac{\pi}{3} \;\Rightarrow\; \theta = \frac{\pi}{6}$. Thus, $D = \{(r,\theta) \mid 1/\sqrt{2} \le r \le \sqrt{\cos 2\theta},\, 0 \le \theta \le \pi/6\}$, so the total area is

$$4A(D) = 4\int_0^{\pi/6}\int_{1/\sqrt{2}}^{\sqrt{\cos 2\theta}} r\,dr\,d\theta = 4\cdot\frac{1}{2}\int_0^{\pi/6}\left[r^2\right]_{r=1/\sqrt{2}}^{r=\sqrt{\cos 2\theta}}d\theta = 2\int_0^{\pi/6}\left[\cos 2\theta - \frac{1}{2}\right]d\theta$$

$$= 2\left[\tfrac{1}{2}\sin 2\theta - \tfrac{\theta}{2}\right]_0^{\pi/6} = \tfrac{\sqrt{3}}{2} - \tfrac{\pi}{6}$$

32. Use polar coordinates to find the volume of the given solid.

Inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the cylinder $x^2 + y^2 = 4$.

**Solution:**

The sphere $x^2 + y^2 + z^2 = 16$ intersects the $xy$-plane in the circle $x^2 + y^2 = 16$, so

$$V = 2 \iint_{4 \le x^2+y^2 \le 16} \sqrt{16 - x^2 - y^2}\, dA \quad \text{[by symmetry]} \quad = 2\int_0^{2\pi} \int_2^4 \sqrt{16 - r^2}\; r\, dr\, d\theta$$

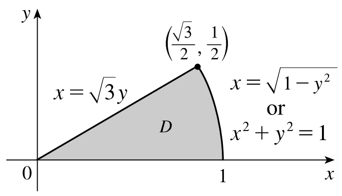
$$= 2\int_0^{2\pi} d\theta \int_2^4 r(16 - r^2)^{1/2} dr = 2\big[\,\theta\,\big]_0^{2\pi} \left[-\tfrac{1}{3}(16 - r^2)^{3/2}\right]_2^4$$

$$= -\tfrac{2}{3}(2\pi)(0 - 12^{3/2}) = \tfrac{4\pi}{3}\left(12\sqrt{12}\right) = 32\sqrt{3}\,\pi$$

41. Evaluate the iterated integral by converting to polar coordinates. $\displaystyle\int_0^{\frac{1}{2}} \int_{\sqrt{3}y}^{\sqrt{1-y^2}} xy^2\, dxdy$

**Solution:**

The region $D$ of integration is shown in the figure. In polar coordinates the line $x = \sqrt{3}\,y$ is $\theta = \pi/6$, so




$$\int_0^{1/2} \int_{\sqrt{3}\,y}^{\sqrt{1-y^2}} xy^2\, dx\, dy = \int_0^{\pi/6} \int_0^1 (r\cos\theta)(r\sin\theta)^2\, r\, dr\, d\theta$$

$$= \textstyle\int_0^{\pi/6} \sin^2\theta\cos\theta\, d\theta \int_0^1 r^4\, dr$$

$$= \left[\tfrac{1}{3}\sin^3\theta\right]_0^{\pi/6} \left[\tfrac{1}{5}r^5\right]_0^1$$

$$= \left[\tfrac{1}{3}\left(\tfrac{1}{2}\right)^3 - 0\right]\left[\tfrac{1}{5} - 0\right] = \tfrac{1}{120}$$