Section 10.2 Calculus with Parametric Curves

18. Find dy/dx and d^2y/dx^2. For which values of t is the curve concave upward?

$$x = t^2 + 1, \qquad y = e^t - 1$$

Solution:

$$x = t^2 + 1,\ y = e^t - 1 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{e^t}{2t} \quad \Rightarrow \quad \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \frac{\frac{2te^t - e^t \cdot 2}{(2t)^2}}{2t} = \frac{2e^t(t-1)}{(2t)^3} = \frac{e^t(t-1)}{4t^3}.$$

The curve is CU when $\frac{d^2y}{dx^2} > 0$, that is, when $t < 0$ or $t > 1$.

31. (a) Find the slope of the tangent line to the trochoid $x = r\theta - d\sin\theta$, $y = r - d\cos\theta$ in terms of θ. (See Exercise 10.1.49.)

(b) Show that if $d < r$, then the trochoid does not have a vertical tangent.

Solution:

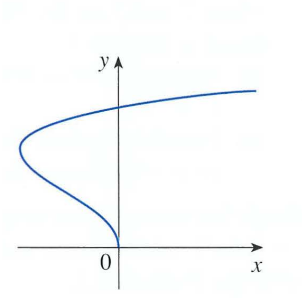
$x = r\theta - d\sin\theta,\ y = r - d\cos\theta.$

(a) $\frac{dx}{d\theta} = r - d\cos\theta$, $\frac{dy}{d\theta} = d\sin\theta$, so $\frac{dy}{dx} = \frac{d\sin\theta}{r - d\cos\theta}$.

(b) If $0 < d < r$, then $|d\cos\theta| \le d < r$, so $r - d\cos\theta \ge r - d > 0$. This shows that $dx/d\theta$ never vanishes, so the trochoid can have no vertical tangent if $d < r$.

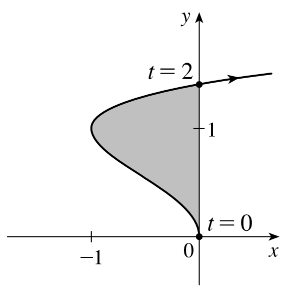
38. Find the area enclosed by the given parametric curve and the y-axis.

$x = t^2 - 2t,$
$y = \sqrt{t}$



Solution:

The curve $x = t^2 - 2t = t(t-2)$, $y = \sqrt{t}$ intersects the y-axis when $x = 0$, that is, when $t = 0$ and $t = 2$. The corresponding values of y are 0 and $\sqrt{2}$. The shaded area is given by



$$\int_{y=0}^{y=\sqrt{2}} (x_R - x_L)\,dy = \int_{t=0}^{t=2} [0 - x(t)]\,y'(t)\,dt = -\int_0^2 (t^2 - 2t)\left(\frac{1}{2\sqrt{t}}\,dt\right)$$

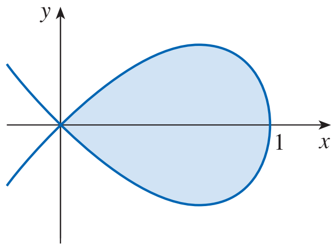
$$= -\int_0^2 \left(\tfrac{1}{2}t^{3/2} - t^{1/2}\right) dt = -\left[\tfrac{1}{5}t^{5/2} - \tfrac{2}{3}t^{3/2}\right]_0^2$$

$$= -\left(\tfrac{1}{5} \cdot 2^{5/2} - \tfrac{2}{3} \cdot 2^{3/2}\right) = -2^{1/2}\left(\tfrac{4}{5} - \tfrac{4}{3}\right)$$

$$= -\sqrt{2}\left(-\tfrac{8}{15}\right) = \tfrac{8}{15}\sqrt{2}$$

40. Find the area of the region enclosed by the loop of the curve

$$x = 1 - t^2, \quad y = t - t^3$$



Solution:

By symmetry, the area of the shaded region is twice the area of the shaded portion above the x-axis. The top half of the loop is described by $x = 1 - t^2$, $y = t - t^3 = t(1-t)(1+t)$, $0 \le t \le 1$ with x-intercepts 0 and 1 corresponding to $t = 1$ and $t = 0$, respectively. Thus, the area of the shaded region is

$2\int_0^1 y\,dx = 2\int_1^0 y(t)\,x'(t)\,dt = 2\int_1^0 (t - t^3)(-2t)\,dt = 4\int_0^1 (t^2 - t^4)\,dt = 4\left[\frac{1}{3}t^3 - \frac{1}{5}t^5\right]_0^1 = 4\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{8}{15}.$

50. Find the exact length of the curve.

$$x = 3\cos t - \cos 3t, \qquad y = 3\sin t - \sin 3t, \; 0 \le t \le \pi$$

Solution:

$x = 3\cos t - \cos 3t,\; y = 3\sin t - \sin 3t,\; 0 \le t \le \pi.\quad \dfrac{dx}{dt} = -3\sin t + 3\sin 3t$ and $\dfrac{dy}{dt} = 3\cos t - 3\cos 3t$, so

$$\begin{aligned}\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= 9\sin^2 t - 18\sin t\,\sin 3t + 9\sin^2 3t + 9\cos^2 t - 18\cos t\,\cos 3t + 9\cos^2 3t\\ &= 9(\cos^2 t + \sin^2 t) - 18(\cos t\,\cos 3t + \sin t\,\sin 3t) + 9(\cos^2 3t + \sin^2 3t)\\ &= 9(1) - 18\cos(t - 3t) + 9(1) = 18 - 18\cos(-2t) = 18(1 - \cos 2t)\\ &= 18[1 - (1 - 2\sin^2 t)] = 36\sin^2 t.\end{aligned}$$

Thus, $L = \int_0^\pi \sqrt{36\sin^2 t}\,dt = 6\int_0^\pi |\sin t|\,dt = 6\int_0^\pi \sin t\,dt = -6\left[\cos t\right]_0^\pi = -6\,(-1-1) = 12.$