112 模組05-09班 微積分2 期考解答和評分標準

1. Evaluate the integrals.

(a) (10%) $\int_0^1 x^2 \cos(1-x)\ dx$. (b) (10%) $\int \frac{\sqrt{(x+1)(x+3)}}{x+2}\ dx$. (c) (10%) $\int \frac{\ln(x+1)}{x^3}\ dx$.

Solution:

(a) We use integration by parts twice in which we always integrate trigonometric functions and differentiate polynomials.

$$\int_0^1 x^2\cos(1-x)\ dx = -x^2\sin(1-x)\Big|_{x=0}^{x=1} + \int_0^1 2x\sin(1-x)\ dx \qquad \text{(3 pts for integration by parts)}$$

$$= -\sin 0 + 0\cdot\sin 1 + \int_0^1 2x\sin(1-x)\ dx = \int_0^1 2x\sin(1-x)\ dx \qquad \text{(1 pt for evaluating } -x^2\sin(1-x)\Big|_{x=0}^{x=1}\text{)}$$

$$= 2x\cos(1-x)\Big|_{x=0}^{x=1} - \int_0^1 2\cos(1-x)\ dx \qquad \text{(2 pts for intrgation by parts)}$$

$$= 2\cos 0 - 0\cdot\cos 1 - 2\int_0^1 \cos(1-x)\ dx = 2 - 2\int_0^1\cos(1-x)\ dx \qquad \text{(1 pt for evaluating } 2x\cos(1-x)\Big|_{x=0}^{x=1}\text{)}$$

$$= 2 + 2(\sin(1-x))\Big|_{x=0}^{x=1} = 2 - 2\sin 1 \qquad \text{(3 pts for integrating } \cos(1-x) \text{ and the final answer)}$$

(b)

$$\int \frac{\sqrt{(x+1)(x+3)}}{x+2}\ dx = \int \frac{\sqrt{x^2+4x+3}}{x+2}\ dx = \int \frac{\sqrt{(x+2)^2-1}}{x+2}\ dx \qquad \text{(2 pts for completing the square)}$$

$$\overset{u=x+2}{=} \int \frac{\sqrt{u^2-1}}{u}\ du \overset{u=\sec\theta,\ 0\le\theta<\frac{\pi}{2},\ \pi\le\theta<\frac{3}{2}\pi}{=} \int \frac{\tan\theta}{\sec\theta}\sec\theta\tan\theta\ d\theta \qquad \text{(4 pts for the substitutions)}$$

$$= \int \tan^2\theta\ d\theta = \int \sec^2\theta - 1\ d\theta = \tan\theta - \theta + C \qquad \text{(2 pts for integrating } \tan^2\theta\text{)}$$

$$= \sqrt{x^2+4x+3} - \sec^{-1}(x+2) + C \qquad \text{(1 pt for } \tan\theta = \sqrt{x^2+4x+3} \text{ and 1 pt for } \theta = \sec^{-1}(x+2)\text{)}$$

(c) We use integration by part for this question(M1). Let $u = \ln(x+1)$ and $dv = \frac{1}{x^3}dx$, and we find $du = \frac{1}{x+1}dx$ and $v = \frac{-1}{2x^2}$(M2). Therefore, we have

$$\begin{aligned}\int \frac{\ln(x+1)}{x^3}dx &= -\frac{\ln(x+1)}{2x^2} - \int \frac{-1}{2x^2(x+1)}dx\ (uv - \int v du)\ (M3)\\ &= -\frac{\ln(x+1)}{2x^2} - \frac{1}{2}\int \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}dx\ (M4)\\ &= -\frac{\ln(x+1)}{2x^2} - \frac{1}{2}\int \frac{(A+C)x^2 + (A+B)x + B}{x^2(x+1)}dx.\end{aligned}$$

Comparing the coefficients (M5), we solve $A = 1$, $B = -1$, and $C = -1$. Continuing integration process, we have

$$\begin{aligned}\int \frac{-1}{x^2(x+1)}dx &= \int \left(\frac{1}{x} + \frac{-1}{x^2} + \frac{-1}{x+1}\right)dx\\ &= \ln|x| + \frac{1}{x} - \ln|x+1| + C(M6)\end{aligned}$$

Putting everything together, we got the final answer

$$-\frac{\ln(x+1)}{2x^2} - \frac{1}{2}\left(\ln|x| - \frac{1}{x} + \ln|x+1|\right) + C.$$

(M1) Any indication of utilizing integration by parts (1%).

(M2) Correctly establishing u and dv (1%). Using the designated u and dv, accurately evaluating du and v (1%).

(M3) Employing the results from M2, correctly formulating $uv - \int v\,du$ (1%).

(M4) Any evidence of employing partial fractions (1%), along with correctly setting up variables (1%). If a student encounters errors in M2 but $\int v\,du$ still necessitates partial fractions, then the student has the opportunity to earn points in M4 and subsequent steps. Otherwise, the student can receive a maximum of 2% for this question.

(M5) Utilizing the outcome from M4, any sign of comparing the coefficients (1%).

(M6) Using the information from M5, for each integral (three integrals in M6), 0.5% if the result is in a similar format (e.g., $\int \frac{1}{x}\,dx = \ln(x)$ or $\int \frac{-1}{x^2} = \frac{1}{x}$), and an additional 0.5% if the result is correct. If a student makes mistakes in M2, M4, or M5, resulting in their final integrals involving two terms or fewer, the student still earns 1% for each correct integral.

PS. Graders are allowed to grant a student 10% credit directly when the final answer is accurate, even if the student did not provide the computation process.

2. For this problem, suppose that $f(x)$ is a one-to-one differentiable function, $f(0) = 0$, and $g(x) = f^{-1}(x)$ is the inverse function of $f(x)$. Also suppose $f'(x)$ is continuous.

(a) (3%) Evaluate $\int_0^{\sin(\pi/4)} \sin^{-1} y\, dy$.

Solution:

Sol 1. Applying integration by part (M1), we set $u = \sin^{-1}(y)$ and $dv = dy$, and get $du = \frac{1}{1-y^2}dy$ and $v = y$ (M2). It follows

$$\int \sin^{-1}(y)dy = y\sin^{-1}(y) - \int \frac{y}{\sqrt{1-y^2}}dy \text{ (M3)}$$

Then, we use U-sub to solve the remaining integral (M4). Let $u = 1 - y^2$, and so $du = -2ydy$ (M5). Thus, we have

$$\begin{aligned}\int \frac{y}{\sqrt{1-y^2}}dy &= \frac{-1}{2}\int \frac{1}{\sqrt{u}}du \text{ (M6)}\\ &= -\sqrt{u} \text{ (M7)}\\ &= -\sqrt{1-y^2} + C \text{ (M8)}\end{aligned}$$

Thus, we evaluate the definite integral

$$\begin{aligned}\int_0^{\sin(\frac{\pi}{4})} \sin^{-1}(y)dy &= y\sin^{-1}(y)\Big|_0^{\sin(\frac{\pi}{4})} + \sqrt{1-y^2}\Big|_0^{\sin(\frac{\pi}{4})} \text{ (M9)}\\ &= \frac{\sqrt{2}\pi + 4\sqrt{2}}{8} - 1 \text{ (M10)}\end{aligned}$$

Scheme 1:

(3%) Correct answer even if the student did not provide the computation process.

(2%) At most two computation mistakes from M1 to M10.

(1%) M1 (0.5%) + M4 (0.5%)

Sol 2. We use the result of 2.(c) (M1):

$$\begin{aligned}\int_0^{\sin(\frac{\pi}{4})} \sin^{-1}(y)dy &= \sin\left(\frac{\pi}{4}\right)\sin^{-1}\left(\sin\left(\frac{\pi}{4}\right)\right) - \int_0^{\frac{\pi}{4}} \sin(x)dx \text{ (M2)}\\ &= \frac{\sqrt{2}\pi}{8} - \left(-\cos(x)\Big|_0^{\frac{\pi}{4}}\right) \text{ (M3)}\\ &= \frac{\sqrt{2}\pi + 4\sqrt{2}}{8} - 1 \text{ (M4)}\end{aligned}$$

Scheme 2:

(3%) Correct evaluation.

(2%) Make just 1 computation mistake in M2 to M4.

(1%) Successfully carryout 2.(c).

(0.5%) Any indication of utilizing 2.(c).

(b) (3%) Consider the function $H(x) = \int_0^{f(x)} g(y)\, dy$. Use FTC part 1 to find $H'(x)$.

Solution:
Let $G(x) = \int_a^x g(t)dt$ for some a (including 0) in the domain of g (M1). Then, it follows $H(x) = \int_0^{f(x)} g(t)dt = \int_a^{f(x)} g(t)dt - \int_a^0 g(t)dt = G(f(x)) - G(0)$ (M2). Therefore, we have

$$\begin{aligned} H'(x) &= G'(f(x))f'(x) \text{ (M3)} \\ &= g(f(x))f'(x) = xf'(x) \text{ (M4)}. \end{aligned}$$

scheme 3:

(M1) Any demonstration of establishing $G(x)$ as an antiderivative of $g(x)$. If the response is in the form of $G(x) = \int g(x)dx$, it is still considered correct (1%).

(M2) (1%).

(M3) Any indication of utilizing the chain rule (0.5%).

(M4) See $g(f(x)) = x$ (0.5%).

scheme 4:
A student might be directly differentiate integral, i.e. no M1 and M2, then we use the following grade scheme

(2%) See M3.

(1%) See M4.

(c) (3%) Use FTC part 2 to show that

$$H(b) - H(a) = \int_{f(a)}^{f(b)} g(y)\, dy = bf(b) - af(a) - \int_a^b f(x)\, dx.$$

Solution:
Note that $H(x) - H(a) = \int_0^{f(x)} g(t)dt - \int_0^{f(a)} g(t)dt = \int_{f(a)}^{f(x)} g(t)dt$, so $H(b) - H(a) = \int_{f(a)}^{f(b)} g(t)dt$ (M1). The derivative of $H(x) - H(a)$ is $xf'(x)$, so $H(x) - H(a)$ is an antiderivative of $xf'(x)$ (M2), i.e., for some C, $H(x) + C = \int_0^x tf'(t)dt$. Moreover, $H(b) - H(a) = \int_a^b tf'(t)dt$ (M3). We evaluate the definite integral:

$$\int_a^b xf'(x)dx = xf(x)\Big|_a^b - \int_a^b f(x)dx \text{ (M4)}$$

(M1) Any demonstration of establishing M1 using the formula:

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx \text{ (1%)}.$$

(M2) Any indication that the student recognizes $H(x) - H(a)$ as an antiderivative of $xf'(x)$; i.e., the student attempts to apply the result from 2.(b) (0.5%).

(M3) (0.5%).

(M4) Any evidence of applying integration by parts (0.5%). Correctly executing the computation (0.5%). Students do not need to expand $xf(x)\Big|_a^b$.

(d) (6%) We know that $f(x) = x + \tan^{-1} x$ satisfies the requirements. Compute

$$\int_{f(1)}^{f(\sqrt{3})} g(y)\ dy.$$

Solution:

Sol 1. Directly applying 2.(c), we immediately have

$$\begin{aligned}\int_{f(1)}^{f(\sqrt{3})} g(t)dt &= \sqrt{3}f(\sqrt{3}) - f(1) - \int_1^{\sqrt{3}} xdx - \int_1^{\sqrt{3}} \tan^{-1}(x)dx \text{ (M1)}\\ &= 2 + \sqrt{3}\tan^{-1}(\sqrt{3}) - \tan^{-1}(1) - \frac{x^2}{2}\bigg|_1^{\sqrt{3}} - \int_1^{\sqrt{3}} \tan^{-1}(x)dx \text{ (M2)}\\ &= 1 + \sqrt{3}\tan^{-1}(\sqrt{3}) - \tan^{-1}(1) - \int_1^{\sqrt{3}} \tan^{-1}(x)dx.\end{aligned}$$

We then use integration by part (M3) to evaluate the remaining integral. Let $u = \tan^{-1}(x)$ and $dv = dx$, so we have $du = \frac{1}{1+x^2}dx$ and $v = x$ (M4). It follows

$$\begin{aligned}\int \tan^{-1}(x)dx &= x\tan^{-1}(x) - \int \frac{x}{1+x^2}dx \text{ (M5)}\\ &= x\tan^{-1}(x) - \frac{1}{2}\int \frac{1}{u}du \ (u = 1+x^2 \text{ and } du = 2xdx) \text{ (M6)}\\ &= x\tan^{-1}(x) - \frac{1}{2}\ln|1+x^2|.\end{aligned}$$

Thus, we have

$$\int_1^{\sqrt{3}} \tan^{-1}(x)dx = \sqrt{3}\tan^{-1}(\sqrt{3}) - \tan^{-1}(1) - \frac{1}{2}(\ln(4) - \ln(2)).$$

Hence, we continue the original integration, which is

$$\begin{aligned}\int_{f(1)}^{f(\sqrt{3})} g(t)dt &= 1 + \sqrt{3}\tan^{-1}(\sqrt{3}) - \tan^{-1}(1) - \left(\sqrt{3}\tan^{-1}(\sqrt{3}) - \tan^{-1}(1) - \frac{1}{2}(\ln(4) - \ln(2))\right)\\ &= 1 + \frac{\ln(2)}{2} \text{ (M7)}.\end{aligned}$$

(M1) (0.5%).

(M2) A student remember the indefinite integral of x is $\frac{x^2}{2}$ (0.5%).

(M3) Any sign of utilizing integration by part for integrating $\tan^{-1}(x)$ (0.5%).

(M4) Correctly set up u and dv (0.5%); correctly evaluate du and v (0.5%).

(M5) Using the information from M4, and correctly set up $uv - \int vdu$.

(M6) Any indication of utilizing u-sub (0.5%), and correctly evaluating the integral of $\frac{x}{1+x^2}$ (Consider $\ln(x^2+1)$ as a correct answer) (0.5%)

(M7) Correctly evaluate $\sqrt{3}f(\sqrt{3}) - f(1)$ in M1 (0.5%). Correctly evaluate $\int_1^{\sqrt{3}} xdx$ (0.5%). Correctly evaluate $\int_1^{\sqrt{3}} \tan^{-1}(x)$ using their answer of $\int \tan^{-1}(x)dx$

PS. Graders are allowed to grant a student 6% credit directly when the final answer is accurate, even if the student did not provide the computation process.

Sol 2. Apply the conclusion of (b), $\int_{f(1)}^{f(\sqrt{3})} g(t)dt = \int_1^{\sqrt{3}} x + \frac{x}{1+x^2}dx = \frac{x^2}{2} + \frac{1}{2}\ln(1+x^2)\bigg|_1^{\sqrt{3}} = 1 + \frac{\ln(2)}{2}$

3. Alice and Bob are considering the improper integral $\int_0^\infty \frac{x^{-\frac{1}{3}}}{1+x^{\frac{4}{3}}}\,dx$.

(a) (3%) Alice used the comparison test to show that $\int_0^1 \frac{x^{-\frac{1}{3}}}{1+x^{\frac{4}{3}}}\,dx$ is convergent. If Alice compared the integrand with a function $\frac{1}{x^p}$, find a value of p and write down the inequality.

(b) (3%) Bob used the comparison test to show that $\int_1^\infty \frac{x^{-\frac{1}{3}}}{1+x^{\frac{4}{3}}}\,dx$ is convergent. If Bob compared the integrand with a function $\frac{1}{x^q}$, find a value of q and write down the inequality.

(c) (8%) Evaluate the improper integral $\int_0^\infty \frac{x^{-\frac{1}{3}}}{1+x^{\frac{4}{3}}}\,dx$.

Solution:

(a) For $0 < x < 1$, we have

$$0 < \frac{x^{-1/3}}{1+x^{4/3}} = \frac{1}{x^{1/3}+x^{5/3}} < x^{-1/3} \quad (1\%)$$

We also compute that

$$\int_0^1 x^{-1/3}\,dx = \lim_{a\to 0^+}\int_a^1 x^{-1/3}\,dx = \lim_{a\to 0^+}\frac{3}{2}x^{2/3}\Big]_a^1 = \lim_{a\to 0^+}\frac{3}{2}(1-a^{2/3}) = \frac{3}{2} \quad (1\%)$$

So we have $p = \frac{1}{3}$ by comparison test. (1%)

(b) For $x \geq 1$, we have

$$0 < \frac{x^{-1/3}}{1+x^{4/3}} = \frac{1}{x^{1/3}+x^{5/3}} < x^{-5/3} \quad (1\%)$$

We also compute that

$$\int_0^1 x^{-5/3}\,dx = \lim_{b\to\infty}\int_1^b x^{-5/3}\,dx = \lim_{b\to\infty} -\frac{3}{2}x^{-2/3}\Big]_1^b = \lim_{b\to\infty}\frac{3}{2}(1-b^{-2/3}) = \frac{3}{2} \quad (1\%)$$

So we have $p = \frac{5}{3}$ by comparison test. (1%)

(c) Set $x = u^3$, $dx = 3u^2\,du$. Then

$$\int \frac{x^{-1/3}}{1+x^{4/3}}\,dx = \int \frac{3u^2}{u(1+u^4)}\,du = \int \frac{3u}{(1+u^4)}\,du. \quad (2\%)$$

Set $v = u^2$, $dv = 2u\,du$. Then

$$\int \frac{x^{-1/3}}{1+x^{4/3}}\,dx = \frac{3}{2}\int \frac{dv}{1+v^2} = \frac{3}{2}\tan^{-1} v + C = \frac{3}{2}\tan^{-1} x^{2/3} + C. \quad (2\%)$$

So we obtain

$$\int_0^1 \frac{x^{-1/3}}{1+x^{4/3}}\,dx = \lim_{a\to 0^+}\int_a^1 \frac{x^{-1/3}}{1+x^{4/3}}\,dx = \lim_{a\to 0^+}\frac{3}{2}(\tan^{-1} 1 - \tan^{-1} a^{2/3}) = \frac{3\pi}{8} \quad (1.5\%)$$

and

$$\int_1^\infty \frac{x^{-1/3}}{1+x^{4/3}}\,dx = \lim_{b\to\infty}\int_1^b \frac{x^{-1/3}}{1+x^{4/3}}\,dx = \lim_{b\to\infty}\frac{3}{2}(\tan^{-1} b^{2/3} - \tan^{-1} 1) = \frac{3\pi}{8}. \quad (1.5\%)$$

Therefore,

$$\int_0^\infty \frac{x^{-1/3}}{1+x^{4/3}}\,dx = \int_0^1 \frac{x^{-1/3}}{1+x^{4/3}}\,dx + \int_1^\infty \frac{x^{-1/3}}{1+x^{4/3}}\,dx = \frac{3\pi}{4} \quad (1\%)$$

4. A lake is enclosed by the curves $y = \ln(\cos x)$, $y = \ln(\sec x)$, and $x = \dfrac{\pi}{3}$.

(a) (2%) Set up, but do not evaluate, an integral with respect to x representing the area of the lake.

$$\int_{\square}^{\square} \boxed{}\, dx$$

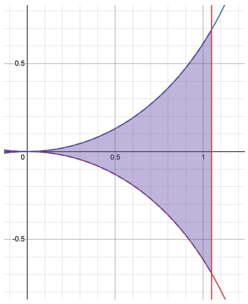
(b) (2%) Set up, but do not evaluate, two integrals with respect to y whose sum represents the area of the lake.

$$\int_{\square}^{\square} \boxed{}\, dy + \int_{\square}^{\square} \boxed{}\, dy$$

(c) (6%) The arc length of a curve described by $y = f(x)$, $a \le x \le b$, is given by the formula

$$\int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

Find the total arc length of the three curves enclosing the lake.



Solution:

(a)

$$\int_{\underline{0}}^{\overline{\pi/3}} (1\%)\, \underline{\ln(\sec x) - \ln(\cos x)}\, (1\%)\, dx.$$

(b)

$$\int_{\underline{0}}^{\overline{\ln 2}} (0.5\%)\, \underline{\frac{\pi}{3} - \sec^{-1} e^y}\, (0.5\%)\, dy + \int_{\underline{-\ln 2}}^{\overline{0}} (0.5\%)\, \underline{\frac{\pi}{3} - \cos^{-1} e^y}\, (0.5\%)\, dy.$$

(c) The total arc length is

$$\begin{aligned}
& \int_0^{\pi/3} \sqrt{1 + ((\ln(\sec x))')^2}\, dx + \int_0^{\pi/3} \sqrt{1 + ((\ln(\cos x))')^2}\, dx \\
+ \quad & \ln(\sec(\frac{\pi}{3})) - \ln(\cos(\frac{\pi}{3}))\ (2\%) \\
= \quad & \int_0^{\pi/3} \sqrt{1 + \tan^2 x}\, dx + \int_0^{\pi/3} \sqrt{1 + \tan^2 x}\, dx + 2\ln 2 = 2\int_0^{\pi/3} \sec x\, dx + 2\ln 2\ (2\%) \\
= \quad & 2\left(\ln|\sec x + \tan x| \Big]_0^{\pi/3} + \ln 2 \right) = 2(\ln(2 + \sqrt{3}) + \ln 2).\ (2\%)
\end{aligned}$$

5. Suppose that X is the random variable that describes the result of a particularly tough exam. The probability density function of X is

$$f_X(x) = \begin{cases} \frac{1}{40} & , \text{for } 9 \le x \le 49, \\ 0 & , \text{for } x < 9 \text{ or } x > 49. \end{cases}$$

(a) (4%) Verify that f_X is a probability density function and show that the expected value (mean) $E(X)$ is 29, where

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\, dx.$$

(b) (3%) Find the variance of X which is

$$\text{Var}(X) = \int_{-\infty}^{\infty} (x - E(X))^2 f_X(x)\, dx.$$

(c) (4%) Let $Y = 10\sqrt{X} + 20$. Follow the steps to find the probability density function of Y, $f_Y(y)$.
For all $50 \le y \le 90$, first define

$$F_Y(y) = \text{Prob}(Y \le y) = \text{Prob}\left(X \le \frac{(y-20)^2}{100}\right).$$

Then $f_Y(y) = \frac{d}{dy} F_Y(y)$. Note that the domain of $f_Y(y)$ is the interval $[50, 90]$.

(d) (4%) Find the expected value (mean) of Y.

Solution:

(a)

$$\int_9^{49} \frac{1}{40}\, dx = 1$$

$$\int_9^{49} \frac{x}{40}\, dx = 29$$

(b)

$$\int_9^{49} \frac{1}{40}(x-29)^2\, dx = \frac{400}{3}$$

(c)

$$f_Y(y) = \frac{d}{dy} \int_9^{\frac{(y-20)^2}{100}} \frac{dx}{40} = \frac{y-20}{2000}$$

Check

$$\int_{50}^{90} \frac{y-20}{2000}\, dy = 1$$

(d)

$$\int_{50}^{90} y \cdot \frac{y-20}{2000}\, dy = \frac{1}{2000}\left[\frac{y^3}{3} - 10y^2\right]_{50}^{90} = \frac{302}{3} - 28 = \frac{218}{3}$$

□

Grading:

- (a) 2% for each verify.
- (b) simple calculation. (-1%) for each mistake.
- (c) Writing the integral is 1%, FTC part 1 is 2%, final answer is 1%.
- (d) This one depends on their answer for (c), but still a simple calculation. (-1%) for each mistake.

6. (a) (8%) A tank that catches runoff from some chemical process initially has 80 L of water with 100 g of pollutant dissolved in it. Polluted water of concentration 5 g/L flows into the tank at a rate of 4 L/hr. The well mixed solution then leaves the tank at the same rate, 4 L/hr. It is known that the amount of pollutant, $P(t)$, satisfies

$$P'(t) = 20 - \frac{P(t)}{20}, \quad P(0) = 100.$$

Find the time when the amount of pollutant reaches 300 g.

(b) (8%) A tank with 80 L of water and 300 g of pollutant is now in the process of diluted draining. The polluted water will be diluted to a concentration of 0.5 g/L and flow into the tank at a rate of 2 L/hr. Meanwhile the outflow rate will be adjusted to 4 L/hr. It is known that the amount of pollutant, $Q(t)$, satisfies

$$Q'(t) = 1 - \frac{4Q(t)}{80-2t}, \quad Q(0) = 300.$$

How much pollutant is in the tank after 20 hr?

Solution:

(a)

$$\int \frac{dP}{400-P} = \int \frac{1}{20}\,dt$$

$$-\ln|400-P| = \frac{t}{20} + C$$

$$P(t) = 400 - Ae^{-t/20}$$

Use $P(0) = 100$ to find $A = 300$.

Solve

$$400 - 300e^{-t/20} = 300$$

$$t = 20\ln 3 \text{ hr}$$

(b)

$$Q'(t) + \frac{2}{40-t}Q(t) = 1$$

$$I(t) = (40-t)^{-2}$$

$$\frac{Q'(t)}{(40-t)^2} + \frac{2Q(t)}{(40-t)^3} = \left(\frac{Q(t)}{(40-t)^2}\right)' = \frac{1}{(40-t)^2}$$

$$Q(t) = (40-t)^2 \int \frac{1}{(40-t)^2}\,dt = 40 - t + C(40-t)^2$$

Use $Q(0) = 300$ to get $C = \frac{13}{80}$.

$$Q(t) = 40 - t + \frac{13}{80}(40-t)^2 = \frac{13}{80}t^2 - 14t + 300$$

$$Q(20) = 85 \text{ g}$$

□

Grading:

- Student can use linear differential equations method for (a).
- 2% each for students showing knowledge on how to solve the differential equation.
- 4% each for solving the differential equation, no simplify needed.
- 2% each for using the initial condition to find the final answer, no simplify needed.
- Each clearly minor mistake is (-1%), each conceptual mistake is (-2%).