1. Let $\mathbf{F}(x,y,z) = e^{-y^2}\mathbf{i} + (-2xye^{-y^2} + e^z)\mathbf{j} + ye^z\mathbf{k}$. You are given that $\mathbf{F}$ is conservative on $\mathbb{R}^3$.
   (a) (3%) Find a scalar potential function of $\mathbf{F}$.
   (b) (3%) Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the line segment from $(0,2,0)$ to $(4,0,3)$.
   (c) (6%) Evaluate $\int_C e^{-y^2}\, dx - 2xye^{-y^2}\, dy + ye^z\, dz$ where C is the line segment in (b).

Solution:

(a)

Assume that $f(x,y,z)$ satisfies

$$f_x = e^{-y^2}\ ,\quad f_y = \left(-2xye^{-y^2} + e^z\right)\ ,\quad f_z = ye^z$$

Then we can first obtain

$$f(x,y,z) = \int f_x\, dx = xe^{-y^2} + g(y,z)$$

Hence

$$f_y = -2xye^{-y^2} + g_y\ ,\quad f_z = g_z \ \Rightarrow\ g_y = e^z,\ g_z = ye^z$$

Next obtain

$$g(y,z) = \int g_y\, dy = ye^z + h(z)$$

Hence

$$g_z = ye^z + h'(z) \ \Rightarrow\ h'(z) = 0$$

Therefore all scalar potential functions are of the form

$$f(x,y,z) = xe^{-y^2} + ye^z + C$$

□

(b)

Method 1: Fundamental Theorem for Line Integrals

$$\int_C \mathbf{F}\cdot\, d\mathbf{r} = f(4,0,3) - f(0,2,0) = 2$$

Method 2: Direct computation

Parametrize the line segment: $\mathbf{r}(t) = \langle 4t, 2-2t, 3t\rangle$, $0 \le t \le 1$.

$$\int_C \mathbf{F}\cdot\, d\mathbf{r} = \int_0^1 \left\langle e^{-(2-2t)^2}, \left(-8t(2-2t)e^{-(2-2t)^2} + e^{3t}\right), (2-2t)e^{3t}\right\rangle\cdot\langle 4,-2,3\rangle\, dt$$

$$= \int_0^1 4e^{-(2-2t)^2} - 2\left(-8t(2-2t)e^{-(2-2t)^2} + e^{3t}\right) + 3(2-2t)e^{3t}\, dt$$

$$= \left[4te^{-(2-2t)^2} + (2-2t)e^{3t}\right]_0^1 = 2$$

□

(c)

Use the parametrization from method 2 in part (b).

Method 1: Fundamental Theorem for Line Integrals

$$\int_C e^{-y^2}\, dx - 2xye^{-y^2}\, dy + ye^z\, dz = \int_C (\mathbf{F} - e^z\mathbf{j})\cdot\, d\mathbf{r}$$

$$= 2 - \int_0^1 -2e^{3t}\, dt = 2 + \frac{2}{3}(e^3 - 1) = \frac{2e^3 + 4}{3}$$

Method 2: Direct computation

$$\int_C e^{-y^2}\, dx - 2xye^{-y^2}\, dy + ye^z\, dz$$

$$= \int_0^1 4e^{-(2-2t)^2} + 16t(2-2t)e^{-(2-2t)^2} + 3(2-2t)e^{3t}\, dt$$

$$= \left[4te^{-(2-2t)^2}\right]_0^1 + 6\int_0^1 (1-t)e^{3t}\ dt$$

$$= 4 + 6\left[\frac{1}{3}(1-t)e^{3t} + \frac{1}{9}e^{3t}\right]_0^1 = 4 + \frac{2}{3}e^3 - 2 - \frac{2}{3} = \frac{2e^3+4}{3}$$

□

Grading:

- Basic distribution: (a) (3%) on finding the potential function, this includes the steps to find/verify the function. (b) (3%) for using the fundamental theorem correctly or evaluating the line integral directly. (c) (2%) for parametrization of the line segment. (2%) for using (b) to evaluate part of the line integral. (2%) for completing the line integral.
- (a) must have justification. (-3%) if no explanation is given. (-1%) if the explanation is incomplete.
- (a) (-2%) immediately if function is wrong, student get points for (b) and (c) as long as they use the same function or compute directly.
- (b) Method 1 MUST use function from (a). Method 2 can only get points if the line integral is successfully evaluated.
- (c) the points for parametrization (2%) is in this part. Evaluating the line integral is (4%) and students can get partial credit for finding ways to use their answer from (a) and (b).
- For everything else, (-1%) for each minor mistake and (-2%) for each concept mistake.

2. Let $\mathbf{F}(x,y) = \ln(1+y)\mathbf{i} + \dfrac{xy}{1+y}\mathbf{j}$.

(a) (8%) Use Green's Theorem to evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$. when C is the boundary of the triangle with vertices $(1,1)$, $(2,1)$, $(1,3)$, oriented counterclockwisely.

(b) (2%) Let $D_\alpha = [0,1]\times[0,\alpha]$ where $\alpha > 0$ be a rectangular region on $\mathbb{R}^2$ and C_α be its boundary, oriented counterclockwisely. Find the value of α that minimizes the line integral $\oint_{C_\alpha} \mathbf{F}\cdot d\mathbf{r}$.

Solution:

(a) Let D be the triangle enclosed by the vertices $(1,1)$, $(2,1)$, and $(1,3)$. Note that the equation for the line passing $(1,3)$ and $(2,1)$ is $y = -2x+5$, or equivalently $x = \dfrac{y-5}{-2}$.

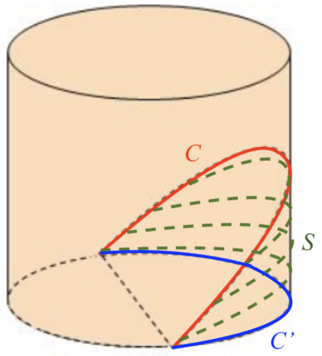
$$
\begin{aligned}
\oint_C \mathbf{F}\cdot d\mathbf{r} &= \oint_C \ln(1+y)\,dx + \frac{xy}{1+y}\,dy \\
&= \iint_D \frac{y}{1+y} - \frac{1}{1+y}\,dA \quad \text{(4 points for Green's theorem)*} \\
&= \iint_D 1\,dA - \iint_D \frac{2}{y+1}\,dA \\
&= 1 - 2\int_1^3 \int_1^{\frac{y-5}{-2}} \frac{1}{y+1}\,dx\,dy \quad \text{(2 points for interated integrals)} \\
&= 1 - 2\int_1^3 \frac{1}{y+1}\left(\frac{y-5}{-2} - 1\right) dy \quad \text{(1 point)} \\
&= 1 + \int_1^3 1 - \frac{4}{y+1}\,dy \\
&= 1 + 2 - 4\ln(4/2) \\
&= 3 - 4\ln 2. \quad \text{(1 point)}
\end{aligned}
$$

*Deduct one point if the sign is not correct.

(b) (1 point) By Green's theorem, we would like to minimize the double integral $\iint_D \dfrac{y-1}{y+1}\,dA$.

(1 point) Note that the integrand $\dfrac{y-1}{y+1} \le 0$ if and only if $-1 < y \le 1$. Thus, when $\alpha = 1$, the integral is minimized.

3. Let $\mathbf{F}(x,y,z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$. Let C be the curve of intersection of the surfaces $x^2+y^2=4$ and $z=y$ above the xy-plane, oriented counterclockwisely when viewed from above.



(a) (7%) Parametrize the curve C and find $\int_C \mathbf{F}\cdot d\mathbf{r}$.

(b) (9%) Let S be part of the cylinder $x^2+y^2=4$ below the curve C and above the xy-plane, oriented away from $(0,0,0)$. Parametrize the surface S and thus evaluate, directly, the flux of curl($\mathbf{F}$) across S :

$$\iint_S \operatorname{curl}(\mathbf{F})\cdot d\mathbf{S}.$$

(c) (4%) Let C' be the curve obtained by projecting C onto the xy-plane, oriented counterclockwisely when viewed from above. By computing $\int_{C'} \mathbf{F}\cdot d\mathbf{r}$, explain how your answers in (a) and (b) are consistent with the Stokes' Theorem.

Solution:

(a) (2M) Parametrize C by $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, 2\sin t\rangle$,
(1M) $0 \le t \le \pi$. Then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^\pi \underbrace{\langle 4\cos t\sin t, 4\sin^2 t, 4\cos t\sin t\rangle}_{(1M)} \cdot \underbrace{\langle -2\sin t, 2\cos t, 2\cos t\rangle}_{(1M)}\,dt$$

$$= \int_0^\pi 8\cos^2 t\sin t\,dt = \underbrace{\frac{16}{3}}_{(2M)}.$$

Grading scheme for 3a.

- (2M) *Correct parametrization for C
- (1M) **Correct range of t
- (1M) Definition of Line integral ($\vec{F}(\vec{r}(t))$)
- (1M) Definition of Line integral ($\vec{r}\,'(t)$)
- (2M) ***Correct answer

Remarks.

(a) If ** is incorrect (say a student wrote $0 \le t \le 2\pi$), no points will be awarded to ***

(b) At most 1M will be deducted overall if a student messed up the orientation of C (and lead to a sign error)

(b) (1M) Parametrize S as $\mathbf{r}(t,z) = \langle 2\cos t, 2\sin t, z\rangle$ where
(1M) $0 \le t \le \pi$,

(1M) $0 \le z \le 2\sin t$.
(1M) Note $\text{curl}(\mathbf{F}) = -y\mathbf{i} - z\mathbf{j} - x\mathbf{k}$ and
(1M) $\mathbf{r}_t \times \mathbf{r}_z = \langle 2\cos t, 2\sin t, 0\rangle$.

$$\begin{aligned}\iint_S \text{curl}\mathbf{F}\cdot d\mathbf{S} &= \int_0^{\pi}\int_0^{2\sin t} \langle -2\sin t, -z, -2\cos t\rangle \cdot \langle 2\cos t, 2\sin t, 0\rangle \, dz\, dt \qquad \text{(2M)}\\ &= \int_0^{\pi}\int_0^{2\sin t} -4\sin t\cos t - 2z\sin t \, dz\, dt\\ &= \int_0^{\pi} -8\sin^2 t\cos t - 4\sin^3 t\, dt\\ &= -\frac{16}{3} \qquad \text{(2M)}\end{aligned}$$

Grading scheme for 3b.

- (1M) (a1) Correct parametrization for S
- (1M) (a2) Correct range of t
- (1M) (a3) Correct range of z
- (1M) (a4) Correct curl
- (1M) (a5) Correct $\mathbf{r}_t \times \mathbf{r}_z$
- (2M) (a6) Converting flux into a double integral
- (2M) (a7) Correct answer

Remarks.

(a) If any of (a1)-(a5) is incorrect, at most 1M can be awarded to (a6), as long as the candidate demonstrates ability to convert a flux integral into a double integral:

$$\iint_S \text{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_D \text{curl}\mathbf{F}(\mathbf{r}(t,z)) \cdot (\mathbf{r}_t \times \mathbf{r}_z)\, dz\, dt$$

However, marks will be taken away from (a6) if students miswrite dA as dS or $d\mathbf{S}$.

(b) At most 1M will be deducted overall if a student messed up the orientation of S (and lead to a sign error)

(c) (1M) Parametrize C' by $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, 0\rangle$, $0 \le t \le \pi$ on xy-plane. Then

$$\text{(1M)} \int_{C'} \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi} \langle 4\cos t\sin t, 0, 0\rangle \cdot \langle -2\sin t, 2\cos t, 0\rangle dt = \int_0^{\pi} -8\cos t\sin^2 t\, dt = 0.$$

By Stokes' Theorem

$$\underbrace{\iint_S \text{curl}\mathbf{F}\cdot d\mathbf{S} = \int_{C'} \mathbf{F}\cdot d\mathbf{r} - \int_C \mathbf{F}\cdot d\mathbf{r}}_{\text{(2M)}} = 0 - \frac{16}{3} = -\frac{16}{3}$$

which equals the answer from (b).

Grading scheme for 3c.

- (1M) Correct parametrization for C'
- (1M) Correct line integral for C'
- (2M) Correct relation of flux of $\text{curl}(\mathbf{F})$ across S with the line integrals along C and C'.

Remarks.
In particular, it is possible to receive full marks in (c) even if (a) and (b) are incorrect.

4. In another universe, the magnitude of the gravitational force $\mathbf{G}$ is inversely proportional to the *cube* of the distance from the origin. In other words,

$$\mathbf{G}(x,y,z) = \frac{K}{(x^2+y^2+z^2)^2}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$$

where K is the gravitational constant. Let $R>0$ and S_R be the sphere $x^2+y^2+z^2=R^2$, oriented outward.

(a) (2%) Explain why the Divergence Theorem cannot be applied to compute the flux of $\mathbf{G}$ across the sphere S_R.

(b) (6%) Find the flux $\iint_{S_R} \mathbf{G}\cdot d\mathbf{S}$. Express your answer in terms of K and R.

(c) (3%) Suppose it is known that $\iint_{S_5} \mathbf{G}\cdot d\mathbf{S} = 8$. Find $\iint_{S_{10}} \mathbf{G}\cdot d\mathbf{S}$.

(d) (6%) Let $U = \{(x,y,z)\in\mathbb{R}^3 : 1 \le x^2+y^2+z^2 \le 4\}$. Compute directly $\iiint_U \operatorname{div}(\mathbf{G})\,\mathrm{d}V$ and explain how the value of this integral and your answer in (b) are consistent with the Divergence Theorem.

Solution:

(a) To apply the Divergence Theorem, $\mathbf{G}$ needs to be C^1in the region enclosed by S_R.
(1M) However, $\mathbf{G}$ is not C^1/ is undefined at $(0,0,0)$ and
(1M) S_R encloses $(0,0,0)$.

Grading scheme for 4a.

- (1M) Pointing out $(0,0,0)$ is a 'problematic point' for $\mathbf{G}$.
- (1M) Pointing out S_R encloses/contains $(0,0,0)$.

(b) (2M) $\mathbf{n} = \dfrac{\langle x,y,z\rangle}{R}$, so
(1M) $\mathbf{G}\cdot\mathbf{n} = \dfrac{K}{R}\cdot\dfrac{1}{x^2+y^2+z^2}$. Hence

$$\text{(3M)} \iint_{S_R} \mathbf{G}\cdot d\mathbf{S} = \frac{K}{R}\iint_{S_R}\frac{1}{x^2+y^2+z^2}\,dS = \frac{K}{R^3}\iint_{S_R} 1\,dS = \frac{K}{R^3}\cdot 4\pi R^2 = \frac{4\pi K}{R}.$$

Grading scheme for 4b.

- (2M) Correct $\mathbf{n}$ (or $\mathbf{r}_\theta\times\mathbf{r}_\varphi$)
- (1M) Simplifying $\mathbf{G}\cdot\mathbf{n}$ or $\mathbf{G}(\mathbf{r}(\theta,\varphi))\cdot(\mathbf{r}_\theta\times\mathbf{r}_\varphi)$.
- (3M) Correct evaluation of the flux integral (Partial credits available)

(c) By (b), we have the outward flux across S_R is inversely proportional to R. Therefore,

$$\text{(2M)} \iint_{S_5} \mathbf{G}\cdot d\mathbf{S} : \iint_{S_{10}} \mathbf{G}\cdot d\mathbf{S} = 10:5$$

Hence, $\iint_{S_{10}} \mathbf{G}\cdot d\mathbf{S} = 4$ (1M).

Grading scheme for 4c.

- (2M) Correct (logically valid) argument explaining the flux across S_5 and S_{10} are in the ratio $2:1$.
- (1M) Correct answer.

(d) (2M) $\operatorname{div}(\mathbf{F}) = -\dfrac{K}{(x^2+y^2+z^2)^2}$.
By spherical coordinates,

$$\iiint_U \operatorname{div}(\mathbf{G})\,dV = \underbrace{\iiint_U -\frac{K}{(x^2+y^2+z^2)^2}\mathrm{d}V = -K\int_0^{2\pi}\int_0^{\pi}\int_1^2 \frac{1}{\rho^4}\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = -2\pi K}_{\text{(2M)}}$$

which equals to $\underbrace{\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} - \iint_{S_1} \mathbf{F} \cdot d\mathbf{S}}_{\text{(2M)}} = \frac{4\pi K}{2} - \frac{4\pi K}{1}$.

This verifies the Divergence Theorem in this case.

Grading scheme for 4d.

- (2M) Correct $\mathbf{div}(\mathbf{F})$ (no partial credits)
- (2M) Correct evaluation of triple integrals (partial credits available)
- (2M) Writing $\iiint_U \operatorname{div}(\mathbf{G})\, dV = \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} - \iint_{S_1} \mathbf{F} \cdot d\mathbf{S}$.

5. For each of the following series, determine whether it is absolutely convergent, conditionally convergent or divergent.

(a) (5%) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$.

(b) (5%) $\sum_{n=1}^{\infty} (-1)^n \cdot \sin\left(\frac{1}{n^2}\right)$.

(c) (6%) $\sum_{n=1}^{\infty} (-1)^n \cdot (e^{\frac{1}{n}} - 1)$.

Solution:

(a) We consider the function $\frac{1}{x \ln x}$. For $x \geq 2$, the function $\frac{1}{x \ln x}$ is continuous, positive, and decreasing (1 point). We see

$$\int_2^{\infty} \frac{1}{x \ln x} dx = \int_{\ln 2}^{\infty} \frac{1}{t} dt = \infty, \text{ with the change of variables } t = \ln x \text{ (2 points).}$$

Therefore, by the integral test, the given series diverges (2 points).

(b) First notice that $0 \leq \frac{1}{n^2} \leq \frac{\pi}{2}$, so $\sin(\frac{1}{n^2}) \geq 0$ and the given series is alternating (1 point) (or one can argue that when n is large, $\sin(\frac{1}{n^2})$ is positive). Apparently, $\sin(\frac{1}{n^2})$ is decreasing to 0 as $n \to \infty$ (1 point), so the given series is convergent by the alternating series test (1 point).

For the absolute convergence, we look at $\sum_{n=1}^{\infty} \sin(\frac{1}{n^2})$. Since $\sin(\frac{1}{n^2})/\frac{1}{n^2} \to 1 > 0$ as $n \to \infty$, the series $\sum_{n=1}^{\infty} \sin(\frac{1}{n^2})$ is convergent by the limit comparison test (1 point). Or one can use the inequality $\sin(\frac{1}{n^2}) \leq \frac{1}{n^2}$ and the fact $\sum \frac{1}{n^2}$ is convergent.

Therefore, the series $\sum_{n=1}^{\infty} (-1)^n \sin(\frac{1}{n^2})$ is absolutely convergent (1 point). Note: one can also just show the absolute convergence.

(c) Notice that $\frac{1}{n} > 0$, so $e^{\frac{1}{n}} - 1 > 0$ and the given series is alternating (1 point). Apparently, $e^{\frac{1}{n}} - 1$ is decreasing to 0 as $n \to \infty$ (1 point), so the given series is convergent by the alternating series test (1 point).

For the absolute convergence, we look at $\sum_{n=1}^{\infty} e^{\frac{1}{n}} - 1$. Since

$$\lim_{n \to \infty} \frac{e^{\frac{1}{n}} - 1}{\frac{1}{n}} = \lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} e^x = 1 > 0 \text{ (2 points)},$$

the series $\sum_{n=1}^{\infty} e^{\frac{1}{n}} - 1$ is divergent by the limit comparison test.

So the series $\sum_{n=1}^{\infty} (-1)^n (e^{\frac{1}{n}} - 1)$ is conditionally convergent (1 point).

6. Let $f(x) = \int_0^x \sqrt{1+t^3}\,\mathrm{d}t$.

(a) (5%) Write down the Maclaurin series for $f(x)$ and specify its radius of convergence. You may express your answer in binomial coefficients $\binom{a}{k}$.

(b) (5%) Find $f^{(7)}(0)$. Express your answer as a rational number $\frac{a}{b}$ with explicit integers a, b.

(c) (5%) Express $f(0.1)$ as an alternating series $b_0 + \sum_{k=1}^{\infty} (-1)^{k-1} b_k$ for some $b_k \geq 0$. Prove that $\{b_k\}_{k=0}^{\infty}$ is a decreasing sequence and find $\lim_{k\to\infty} b_k$.

(d) (2%) Hence, determine how many terms of the series in (c) are needed in order to estimate the value of $f(0.1)$ up to an error of 10^{-5}.

Solution:

(a)

$$f(x) = \int_0^x \sqrt{1+t^3}dt = \int_0^x \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} t^{3n} dt \text{ (1 point)}$$

$$= \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \int_0^x t^{3n}\, dt = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \frac{1}{3n+1} x^{3n+1}. \text{ (2 points)}$$

By the term by term integration theorem, the radius of convergence of $\sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \frac{1}{3n+1} x^{3n+1}$ and the radius of convergence of $\sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} x^{3n}$ are the same. And by the binomial series we know that $\sum \binom{\frac{1}{2}}{n} x^{3n}$ converges if $|x^3| < 1$ and diverges if $|x^3| > 1$ which means that the radius of convergence of $\sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} x^{3n}$ is 1. Hence the radius of convergence of $\sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \frac{1}{3n+1} x^{3n}$ is 1. (2 points)

Or we can use the Ratio Test to find the radius of convergence of $\sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \frac{1}{3n+1} x^{3n+1}$.

Let $a_n = \binom{\frac{1}{2}}{n} \frac{1}{3n+1} x^{3n+1}$.

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{1}{2}-n}{n+1}\right| \cdot \frac{3n+1}{3n+4} \cdot |x|^3 \to |x|^3$ as $n \to \infty$.

Hence $\sum a_n$ converges absolutely if $|x|^3 < 1$ and $\sum a_n$ diverges if $|x|^3 > 1$. Thus the radius of convergence of $\sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \frac{1}{3n+1} x^{3n+1}$ is 1. (2 points)

(b) Since $f(x) = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \frac{1}{3n+1} x^{3n+1} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$, compare coefficients in front of x^7 and we obtain $\binom{\frac{1}{2}}{2} \frac{1}{7} = \frac{f^{(7)}(0)}{7!}$. (2 points)

Hence $f^{(7)}(0) = \binom{\frac{1}{2}}{2} \cdot 6! = \frac{\frac{1}{2}\cdot(-\frac{1}{2})}{2!} 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6 = -90$ (3 points)

Thus $f^{(7)}(0) = -90 = \frac{a}{b}$ where $a = -90$, $b = 1$.

(c) $f(0.1) = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} \frac{1}{3k+1} \frac{1}{10^{3k+1}}$

Note that $\binom{\frac{1}{2}}{k} \frac{1}{3k+1} \frac{1}{10^{3k+1}} = \frac{\frac{1}{2}\cdot(-\frac{1}{2})\cdots(\frac{1}{2}-k+1)}{k!(3k+1)10^{3k+1}}$ and $\binom{\frac{1}{2}}{0} \frac{1}{3\cdot 0+1} \frac{1}{10^{3\cdot 0+1}} = \frac{1}{10}$ (1 point)

Hence $f(0.1) = \frac{1}{10} + \sum_{k=1}^{\infty} (-1)^{k+1} b_k$ where $b_1 = \frac{1}{8} \times \frac{1}{10^4}$ and $b_k = \frac{1\cdot 3\cdot 5\cdots(2k-3)}{2^k k!(3k+1)10^{3k+1}}$, for $k \geq 2$. (1 point)

$\frac{b_{k+1}}{b_k} = \frac{(2k-1)(3k+1)}{2(k+1)(3k+4)10^3} < \frac{1}{10^3} < 1$. Hence $\{b_k\}$ is decreasing. (1 point)

Moreover, $0 < b_k = \frac{1\cdot 3\cdot 5\cdots(2k-3)}{2\cdot 4\cdot 6\cdots(2k-2)\cdot(2k)} \cdot \frac{1}{(3k+1)10^{3k+1}} < \frac{1}{10^{3k+1}}$

Thus $\lim_{k\to\infty} b_k = 0$ by the squeeze theorem. (2 points)

(d) From (c), we can apply alternating series estimation theorem on $f(0.1) = \frac{1}{10} + \sum_{k=1}^{\infty}(-1)^{k+1}b_k$.

Thus $|f(0.1) - (\frac{1}{10} + \sum_{k=1}^{n}(-1)^{k+1}b_k)| < b_{n+1}$ (1 point)

Since $b_2 < \frac{1}{10^5}$, we can use the first two terms $\frac{1}{10} + b_1 = \frac{1}{10} + \frac{1}{80000}$ to estimate $f(0.1)$ and the error is less than 10^{-5}. (1 point)

7. Let $a_n = \dfrac{n^n}{n!}$ and consider the power series $f(x) = \sum_{n=1}^{\infty} a_n x^n$.

(a) (5%) Find the radius of convergence of $f(x)$.

(b) (2%) For $n \geq 2$, we set $s_n = \sum_{k=1}^{n-1} \ln(k)$. By considering the graph of $y = \ln(x)$, interpret s_n as an area and deduce that $s_n < \int_1^n \ln(x)\,\mathrm{d}x$.

(c) (3%) Note that $s_n = \ln((n-1)!)$. Use (b) to prove that $(n-1)! < \left(\dfrac{n}{e}\right)^n \cdot e$.

(d) (4%) Let $b_n = \dfrac{a_n}{e^n}$. Use (c) to determine whether the series $\sum_{n=1}^{\infty} b_n$ converges or not.

(e) (2%) Prove that the sequence $\{b_n\}_{n=1}^{\infty}$ is decreasing.
Hint. You may use, without proof, the fact that $\left(1 + \dfrac{1}{n}\right)^n < e$ for $n > 0$.

(f) (2%) Hence, find the interval of convergence of the power series $f(x)$.
Hint. You may use, without proof, the fact that $\lim_{n\to\infty} b_n = 0$.

Solution:

(a) The ratio of the successive terms is

$$\frac{a_{n+1}x^{n+1}}{a_n x^n} = \frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} x = \left(1 + \frac{1}{n}\right)^n x,$$

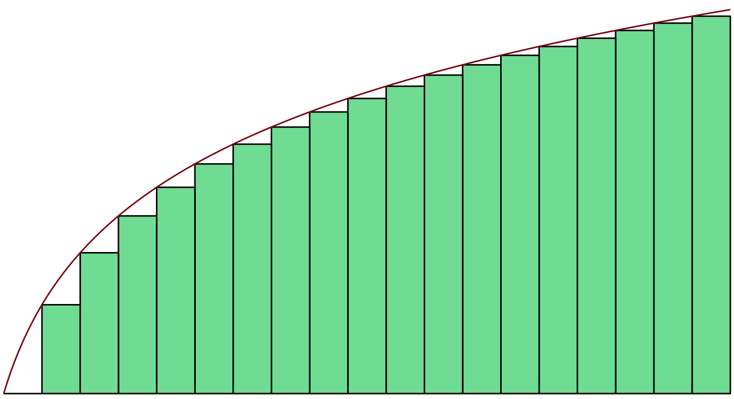
so

$$\lim_{n\to\infty} \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = e|x| < 1$$

gives $|x| < e^{-1}$. Hence, the radius of convergence of $f(x)$ is e^{-1}.

- 1% for considering the ratio of the successive terms
- 1% for simplification of the ratio
- 1% for the considering the limit of the ratio
- 1% for getting the correct limit
- 1% for concluding the correct radius of convergence

(b) Because $\ln x$ is monotonically increasing, s_n is interpreted as the total area of the green rectangles below the graph of $y = \ln x$ (see the figure on the next page):



Hence,

$$s_n < \int_1^n \ln x \, dx.$$

- 1% for sketching the rectangles with the graph
- 1% for the approximately correct shape of the graph of $y = \ln x$

(c) Since

$$s_n = \ln\left(\prod_{k=1}^{n-1} k\right) = \ln\left((n-1)!\right)$$

and

$$\int_1^n \ln x\, dx = [x\ln x - x]_1^n = n(\ln n - 1) + 1,$$

we obtain

$$(n-1)! < e^{n(\ln n - 1)+1} = \left(\frac{n}{e}\right)^n \cdot e$$

by exponentiating both sides of the inequality of (b).

- 1% for correctly getting the anti-derivative of $\ln x$
- 1% for correctly getting $\int_1^n \ln x\, dx$
- 1% for deducing the conclusion by correctly exponentiating the equation of (b)

(d) By (c), we get

$$\frac{1}{n} = \frac{(n-1)!}{n!} < \frac{\left(\frac{n}{e}\right)^n \cdot e}{n!} = e \cdot b_n,$$

Since

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is divergent,

$$\sum_{n=1}^{\infty} b_n$$

is also divergent by Comparison Test.

- 1% for the trial of using the Comparison Test
- 1% for deducing the inequality $\frac{1}{n} < e \cdot b_n$ or equivalent useful inequality
- 1% for stating $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent
- 1% for the correct use of Comparison Test hence deducing the conclusion

(e) The ratio of the successive two terms of the sequence $\{b_n\}_{n=1}^{\infty}$ is

$$\frac{b_{n+1}}{b_n} = \frac{a_{n+1}}{a_n e} = \frac{1}{e}\left(1 + \frac{1}{n}\right)^n < 1,$$

Since $b_n > 0$ for $n \geq 1$, this implies $b_{n+1} < b_n$, hence $\{b_n\}_{n=1}^{\infty}$ is decreasing.

- 1% for a trial of ratio test for b_n
- 1% for obtaining the inequality $\frac{b_{n+1}}{b_n} < 1$

(f) The radius of convergence of $f(x)$ is $\frac{1}{e}$ by (a), and at the boundaries $x = \pm\frac{1}{e}$,

$$f\left(\frac{1}{e}\right) = \sum_{n=1}^{\infty} b_n$$

is divergent by (d) and

$$f\left(-\frac{1}{e}\right) = \sum_{n=1}^{\infty} (-1)^n b_n$$

is convergent by Alternating Series Test since $\{b_n\}_{n=1}^{\infty}$ is decreasing by (e) and $\lim_{n\to\infty} b_n = 0$. Therefore, the interval of convergence of $f(x)$ is $\left[-\frac{1}{e}, \frac{1}{e}\right)$.

- 1% for the correct application of the Alternating Series Test to $f\left(-\frac{1}{e}\right)$ to prove its convergence
- 1% for arriving at the correct conclusion