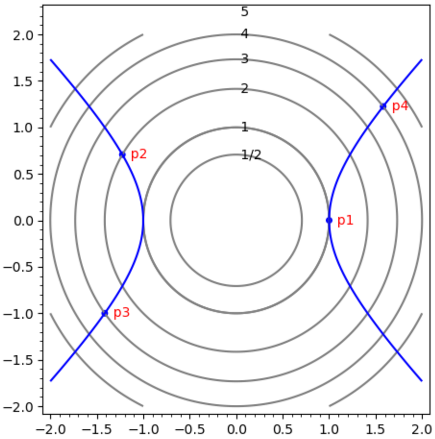
1. In the following diagram, the circles are the level curves of the function $f(x,y)$, and the hyperbola is given by $g(x,y)=1$. Circle the best answer.

在下圖中，圓形為函數 $f(x,y)$ 的等高線，而雙曲線則為 $g(x,y)=1$ 的圖形。請圈選最佳答案。



(a) (3%) It is known that the function $f(x,y)$ attains its minimum value subject to $g(x,y)=1$ at one of the points p_1, p_2, p_3, p_4. Which one should it be ?

已知函數 $f(x,y)$ 於 $g(x,y)=1$ 上的 p_1, p_2, p_3, p_4 其中一點達到其最小值。 應該是哪一個點呢?

p_1 / p_2 / p_3 / p_4

(b) (4%) In the diagram above, draw clearly the gradient vector of the function $f(x,y)$ at p_2. Credits will be given if the gradient vector is drawn towards the **correct direction**.

請在上圖中清楚畫出 $f(x,y)$ 在 p_2 的梯度向量，你必須正確表示梯度向量的方向才能得到所有分數。

(c) (3%) At p_2, decide if the directional derivative of $f(x,y)$ in the direction $\mathbf{u}$ is positive, negative, or zero.

在點 p_2，判斷 $f(x,y)$ 於下列方向 $\mathbf{u}$ 的方向導數為正、負、還是零。

(i) $\mathbf{u}=\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$. Positive 正 / Negative 負 / Zero 零

(ii) $\mathbf{u}=\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$. Positive 正 / Negative 負 / Zero 零

(iii) $\mathbf{u}=\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$. Positive 正 / Negative 負 / Zero 零

Solution:

(a) p_1 (3M)

(b) An arrow perpendicular to the circle at p_2 and pointing at the northwest direction.
2M for Normality and 2M for Direction.

(c) (i) Positive (1M)
(ii) Zero (1M)
(iii) Negative (1M)

2. Let 考慮函數 $f(x,y) = xy - \sqrt{3+2x^2-y^2}$.

(a) (4%) Find 求 $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

(b) (6%) Find an equation of the tangent plane to the graph $z = f(x,y)$ when $x = 1$ and $y = -1$.
求曲面 $z = f(x,y)$ 當 $x = 1$ 和 $y = -1$ 時的切面方程式。

(c) (4%) Use the linearization of $f(x,y)$ at $(1,-1)$ to estimate the value of $f(0.97,-1.02)$.
使用 $f(x,y)$ 在 $(1,-1)$ 處的線性逼近來估計 $f(0.97,-1.02)$ 的值。

Solution:

(a)

$$f_x = y - \frac{2x}{\sqrt{3+2x^2-y^2}}$$

$$f_y = x + \frac{y}{\sqrt{3+2x^2-y^2}}$$

□

(b) Method 1:

$$f(1,-1) = -3 \qquad f_x(1,-1) = -2 \qquad f_y(1,-1) = \frac{1}{2}$$

Tangent plane formula: $z = -3 - 2(x-1) + \dfrac{1}{2}(y+1)$ □

Method 2: $z = xy - \sqrt{3+2x^2-y^2}$ can be re-written as $(xy-z)^2 - 3 - 2x^2 + y^2 = 0$. Let $F(x,y,z) = (xy-z)^2 - 3 - 2x^2 + y^2$.

$$\nabla F = \langle 2y(xy-z) - 4x, 2x(xy-z) + 2y, -2(xy-z) \rangle$$

$f(1,-1) = -3$, $F(1,-1,-3) = 0$.

$$\nabla F(1,-1,-3) = \langle -8, 2, -4 \rangle$$

Tangent plane formula: $-8(x-1) + 2(y+1) - 4(z+3) = 0$ □

(c)

$$L(x,y) = -3 - 2(x-1) + \frac{1}{2}(y+1)$$

$$f(0.97,-1.02) \approx L(0.97,-1.02) = -3 + 0.06 - 0.01 = -2.95$$

□

Grading guideline:

(-2%) for minor mistakes and (-4%) for conceptual mistakes.

Parts (b) and (c) should use their answer in (a). Part (c) should use their answer in (b). Do not take points off for follow through on (b) and (c).

3. Suppose that the function $f(x,y) = 4x^2y + x^2 + 2xy^2 - y + 100$ describes the concentration of bacteria at the point (x,y) in the NTU Ecological Pond. You are sitting in a boat at the point $P = (2,1)$.
假設函數 $f(x,y) = 4x^2y + x^2 + 2xy^2 - y + 100$ **描述台大醉月湖中點** (x,y) **的細菌濃度。現在，你的船在湖中** $P = (2,1)$ **這個點上。**

(a) (5%) What is the rate of change of the concentration of bacteria in the northwest direction $\mathbf{u} = (-1,1)$ at P?
在P**點上的西北方向** $\mathbf{u} = (-1,1)$ **細菌濃度的變化率為多少？**

(b) (4%) At P, find the maximum rate of the change of the concentration of bacteria.
求在 P **點上細菌濃度最大的變化率。**

(c) (5%) A scientist finds out that the bacteria seems to have a tendency to grow in the east : $f(x,y)$ is indeed a function in time t with $x = t^3 + t$ and $y = t\ln t + 1$. Find $\left.\dfrac{df}{dt}\right|_{t=1}$.
科學家發現細菌似乎有在東面生長的趨勢：$f(x,y)$ **是時間** t **的函數，其中** $x = t^3 + t$ **和** $y = t\ln t + 1$**。求** $\left.\dfrac{df}{dt}\right|_{t=1}$**。**

Solution:

(a) The gradient is $\nabla f(2,1) = \underbrace{\langle 8xy + 2x + 2y^2,\, 4x^2 + 4xy - 1\rangle}_{\text{1M+1M}}|_{(2,1)} = \underbrace{\langle 22, 23\rangle}_{\text{1M}}$.

The direction $\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle$. (1M)

The required rate of change is

$$\underbrace{\nabla f(2,1)\cdot\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle}_{\text{1M}} = \underbrace{\frac{1}{\sqrt{2}}}_{\text{1M}}.$$

Grading scheme of Q3a.

- 1M+1M for correct f_x and f_y
- 1M for correct $\nabla f(2,1)$
- 1M for the formula 'gradient dot direction'
- 1M for correct answer

(b) (2M) The greatest rate of change at P is given by $\|\nabla f(P)\|$,
(2M) which equals $\sqrt{(22)^2 + (23)^2}$.

Grading scheme of Q3b.

- 2M for knowing the max. rate equals the length of the gradient vector
- 2M for correct answer (1M for minor mistakes such as forgetting to take square root, etc)

(c) By chain rule, we have

$$\begin{aligned} f_t &= f_x \cdot x_t + f_y \cdot y_t \qquad \text{(1M)} \\ &= f_x(x,y)(3t^2+1) + f_y(x,y)(\ln t + 1) \qquad \text{(1M)} \end{aligned}$$

When $t = 1$, we have $x = 2$ and $y = 1$ (1M) and so

$$f_t(1) = f_x(2,1)(4) + f_y(2,1) = 22\cdot 4 + 23 = 111 \qquad ((2M)).$$

Grading scheme of Q3c.

- 1M for knowing the chain rule $f_t = f_x \cdot x_t + f_y \cdot y_t$
- 1M for correct derivatives x_t and y_t
- 1M for knowing $x(1) = 2$ and $y(1) = 1$
- 2M for the correct answer (1M for clear minor calculation mistake)

4. Let $f(x,y) = x^3 - \alpha x + \beta xy^2 + y^3$. You are given that $\left(-\frac{1}{\sqrt{15}}, \frac{2}{\sqrt{15}}\right)$ is one of its critical points.

考慮函數 $f(x,y) = x^3 - \alpha x + \beta xy^2 + y^3$**，已知** $\left(-\frac{1}{\sqrt{15}}, \frac{2}{\sqrt{15}}\right)$ **為其中一個臨界點。**

(a) (4%) Find the values of α and β.
求 α **和** β **的值。**

(b) (10%) Hence, find all the critical points of $f(x,y)$ and classify them into either local maximum, local minimum or saddle point.
由此找出 $f(x,y)$ **的所有臨界點，並逐一判斷他們為局部最大值、局部最小值、或是鞍點。**

Solution:

Explain grading scheme: a* - give * points if you see correct answers; m1 - give 1 point if you see any sign of applying a method.

(a) Set a system of equation

$$\begin{cases} \frac{\partial f}{\partial x}\left(-\frac{1}{\sqrt{15}}, \frac{2}{\sqrt{15}}\right) & = 0 \\ \frac{\partial f}{\partial y}\left(-\frac{1}{\sqrt{15}}, \frac{2}{\sqrt{15}}\right) & = 0 \end{cases} \Rightarrow \begin{cases} \frac{1}{5} - \alpha + \frac{4}{15}\beta = 0 \\ -\frac{1}{15}\beta + \frac{3}{15} = 0 \end{cases}.$$

Therefore, $\alpha = 1$ and $\beta = 3$.

Grading scheme:

- m1 Any sign of setting the system of equation.
- a1 Correctly evaluate $\partial f/\partial x$.
- a1 Correctly evaluate $\partial f/\partial y$.
- a1 Correctly solve α and β.

(b) Set a system of equation

$$\nabla f(x,y) = (0,0) \Rightarrow \begin{cases} 3x^2 - 1 + 3y^2 = 0 \\ 2xy + y^2 = 0 \end{cases}.$$

The second equation implies $2x + y = 0$ or $y = 0$. The first equation deduces the critical points $\left(-\frac{1}{\sqrt{15}}, \frac{2}{\sqrt{15}}\right)$, and $y = 0$ implies $x = \pm\frac{1}{\sqrt{3}}$.

Let us compute the discriminant. We have

$$\Delta(x,y) = \det\begin{pmatrix} 6x & 6y \\ 6y & 6x + 6y \end{pmatrix} = 36(x^2 + xy - y^2).$$

Evaluate the discriminant at each critical point.

- For $\left(-\frac{1}{\sqrt{15}}, \frac{2}{\sqrt{15}}\right)$, $\Delta < 0$. Therefore, $\left(-\frac{1}{\sqrt{15}}, \frac{2}{\sqrt{15}}\right)$ is a *saddle point.*
- For $\left(\pm\frac{1}{\sqrt{3}}\right)$, $\Delta > 0$. Therefore, $\left(\frac{1}{\sqrt{3}}, 0\right)$ is a local minimal, and $\left(-\frac{1}{\sqrt{3}}, 0\right)$ is a local maximal by evaluate f_{xx}.

Grading scheme: Allow students to use any answer they got in the previous question.

- m1 Setting $\nabla f = 0$.
- a1 Correctly setting up the system of equations.
- m1 Any sign of attempting to solve the system of equations.
- a1 Correctly solve the system of equations.
- m1 Attempting to find the discriminant.
- a1 Finding the correct discriminant.
- a1 Correctly evaluating the discriminant at each point.
- a3 Correctly applying the second order test. 1 point for each correct answer.

5. (14%) Use the method of Lagrange multipliers to find the maximum and minimum values of $f(x,y,z) = x^2yz^4$ subject to the constraint $9x^2 + y^2 + 4z^2 = 49$.
使用 Lagrange 乘子法求函數 $f(x,y,z) = x^2yz^4$ 在限制條件 $9x^2 + y^2 + 4z^2 = 49$ 下的最大值和最小值。

Solution:

The constraint is an ellipsoid, which is a closed and bounded region. By the extreme value theorem, both the maximum and the minimum of the function f occur at the critical points.

Let $g(x,y,z) = 9x^2 + y^2 + 4z^2$. Then the system of equations we need to solve is $\nabla f = \lambda \nabla g$, $g = 49$.

$$2xyz^4 = 18x\lambda$$
$$x^2z^4 = 2y\lambda$$
$$4x^2yz^3 = 8z\lambda$$
$$9x^2 + y^2 + 4z^2 = 49$$

Hence we can get

$$4x^2yz^4 = 36x^2\lambda = 8y^2\lambda = 8z^2\lambda$$

and we divide into 2 cases: $\lambda \neq 0$ and $\lambda = 0$.

For $\lambda \neq 0$, we have $9x^2 = 2y^2 = 2z^2$. Plug in to get $7y^2 = 49$. The critical points will be $\left(\pm\sqrt{14/9}, \pm\sqrt{7}, \pm\sqrt{7}\right)$.

For $\lambda = 0$, we know that $x^2yz^4 = 0$.

Therefore the maximum value of f is $\frac{2 \cdot 7^3}{9}\sqrt{7}$ and the minimum value of f is $-\frac{2 \cdot 7^3}{9}\sqrt{7}$ □

Grading guideline:

(6%) for the system of equations: (2%) for finding the gradient vector for the functions and (4%) for understanding the method of Lagrange multipliers.

(6%) for solving the system: (-2%) for each missing case along the steps.

Example: $2xyz^4 = 18x\lambda \Rightarrow yz^4 = 9\lambda$. Need to consider $x = 0$.

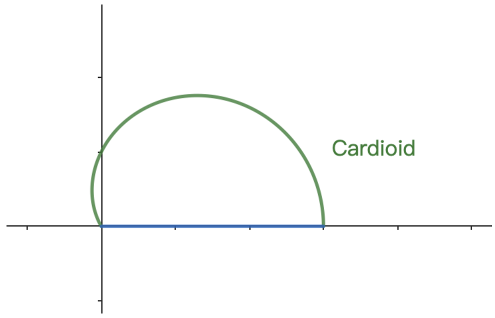
(2%) for evaluating the maximum and minimum values. Students can get points for this even if they made mistakes above.

(-1%) for each minor mistake (miscopy, miscalculate, misread). Conceptual mistakes would be at least (-2%) each.

6. (a) (10%) Evaluate 求 $\int_0^4 \int_{\sqrt{y}}^2 \cos(x^3)\,dx\,dy$.

(b) (10%) Find the area enclosed by the part of the cardioid $r = 1 + 2\cos\theta$, $0 \le \theta \le \frac{2\pi}{3}$ and the x-axis.

求心臟線 $r = 1 + 2\cos\theta$, $0 \le \theta \le \frac{2\pi}{3}$ 和 x-軸圍成的範圍的面積。



Solution:

Explain grading scheme: a* - give * points if you see correct answers; m1 - give 1 point if you see any sign of applying a method.

(a) The area $\Omega : 0 < y < 4,\ \sqrt{y} < x < 2$ is equivalent to the following

$$\Omega : 0 < x < 2,\ 0 < y < x^2.$$

Therefore, we have

$$\begin{aligned}\int_0^4 \int_{\sqrt{y}}^2 \cos(x^3)dxdy &= \int_0^2 \int_0^{x^2} cos(x^3)dydx \\ &= \int_0^2 x^2\cos(x^3)dx.\end{aligned}$$

Let $u = x^3$, so $du = 3x^2dx$. The indefinite integral of the last integral above is

$$\int x^2\cos(x^3)dx = \frac{1}{3}\int \cos(u)du = \frac{\sin(u)}{3} = \frac{\sin(x^3)}{3}.$$

Hence, the integral is

$$\int_0^2 x^2\cos(x^3)dx = \left.\frac{\sin(x^3)}{3}\right|_0^2 = \frac{\sin(8)}{3}.$$

(Another approach) Let $u = x^3$, so $du = 3x^2dx$ and $0 < u < 8$. Therefore, the definite integral is

$$\int_0^2 x^2\cos(x^3)dx = \frac{1}{3}\int_0^8 \cos(u)du = \left.\frac{\sin(u)}{3}\right|_0^8 = \frac{\sin(8)}{3}.$$

Grading scheme:

- m1 Any sign of swapping the order of x and y for Ω.
- a4 Correctly giving the limits of Ω. 1 point for each limit.
- m1 Swapping the order of integration.
- a1 Correctly evaluating the most inner integral.
- m1 Any sign of applying the change of variable for the outer integral.
- a1 Correctly finding the indefinite integral.
- a1 Correctly evaluating the definite integral.

The last 2 points can be change to if students approach from the second method:

- a1 Correctly finding the limits of u.
- a1 Correctly finding the indefinite integral, i.e. see $\frac{\sin(u)}{3}$.

(b) We will evaluate

$$\int_0^{2\pi/3}\int_0^{1+2\cos\theta} r dr d\theta.$$

It is

$$\begin{aligned}
\frac{1}{2}\int_0^{2\pi/3}(1+2\cos\theta)^2 d\theta &= \frac{1}{2}\int_0^{2\pi/3} 1+4\cos\theta+4\cos^2\theta d\theta \\
&= 2\int_0^{2\pi/3}\cos\theta d\theta + \frac{1}{2}\int_0^{2\pi/3} 3-2+4\cos^2\theta d\theta \\
&= 2\sin\theta + \frac{3}{2}\theta\Big|_0^{2\pi/3} + \frac{1}{2}\int_0^{2\pi/3}(2\cos^2\theta-1)d\theta \\
&= \sqrt{3}+\pi+\int_0^{2\pi/3}\cos(2\theta)d\theta \\
&= \sqrt{3}+\pi+\frac{1}{2}\sin(2\theta)\Big|_0^{2\pi/3} = \frac{3\sqrt{3}}{4}+\pi
\end{aligned}$$

Grading Scheme:

- m1 See $\int\int r dr d\theta$.
- a4 Correct limits for the double integral. 1 point for each limit.
- a1 Correctly find the most inner "indefinite" integral.
- m1 Any sign of applying double angle formula.
- a1 Correctly applying double angle formula.
- a2 Correctly evaluate the double integral. Get a1 if the entire process is correct with some minor computation error.

7. (14%) Evaluate 求

$$\iint_D \sqrt{x}y^2\,dA$$

where D is enclosed by $xy = 1$, $xy = 8$, $y = \sqrt{x}$, $y = 5\sqrt{x}$ in the first quadrant.
其中 D 為 $xy = 1$, $xy = 8$, $y = \sqrt{x}$, $y = 5\sqrt{x}$ 在第一象限圍成的區域。

Solution:

(2M) Let $u = xy$ and $v = \dfrac{y}{\sqrt{x}}$ (2M).

(2M) Then $x = (u/v)^{2/3}$ and $y = u^{1/3}v^{2/3}$.

(2M) The Jacobian equals

$$= \begin{vmatrix} \frac{2}{3}u^{-1/3}v^{-2/3} & -\frac{2}{3}u^{2/3}v^{-5/3} \\ \frac{1}{3}u^{-2/3}v^{2/3} & \frac{2}{3}u^{1/3}v^{-1/3} \end{vmatrix} = \frac{2}{3v}.$$

(2M) The region becomes a rectangle $1 \le u \le 8$ and $1 \le v \le 5$. Therefore,

$$\begin{aligned} \iint_D \sqrt{x}y^2\,dA &= \underbrace{\int_1^5 \int_1^8 uv \cdot \frac{2}{3v}\,du\,dv}_{3\text{M}} \\ &= \underbrace{\int_1^5 \left[\frac{u^2}{3}\right]_{u=1}^{u=8} dv}_{1\text{M}} \\ &= \int_1^5 21\,dv \\ &= \underbrace{84}_{2\text{M}} \end{aligned}$$

Marking scheme.

(2M)	Making a suitable substitution $u = u(x, y)$, $v = v(x, y)$
(2M)	Solving for $x = x(u, v)$ and $y = y(u, v)$
(2M)	Correct Jacobian
(2M)	Correct transformed region
(3M)	Correctly transformed the integral (integrand, integration limits)
(1M)	Correct anti-derivative for du or dv
(2M)	Correct answer