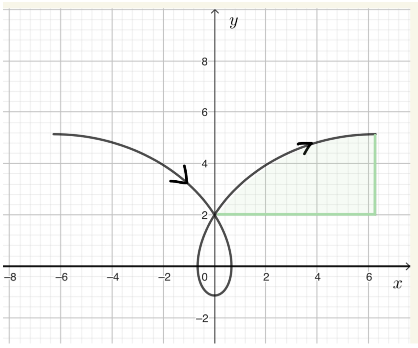
1. (20 %) A curve C is defined by the parametric equations $x = 2t - \pi \sin(t)$, $y = 2 - \pi \cos(t)$, where $-\pi \le t \le \pi$.

   (a) (4 %) Find $\dfrac{dy}{dx}$.

   (b) (4 %) Show that C has two tangents at the point $(x, y) = (0, 2)$ and find the equations of these tangent lines.

   (c) (5 %) Find $\dfrac{d^2y}{dx^2}$. Is C concave upward or downward near $t = \dfrac{\pi}{3}$?

   (d) (7 %) Find the area of shaded region which is enclosed by the curve C, $x = 2\pi$ and $y = 2$.



Solution:

(a)

$$\begin{aligned} \frac{dy}{dt} &= \pi \sin(t) \text{ (1 point)} \\ \frac{dx}{dt} &= 2 - \pi \cos(t) \text{ (1 point)} \\ \frac{dy}{dx} &= \frac{\pi \sin(t)}{2 - \pi \cos t} \text{ (2 points).} \end{aligned}$$

(b) We want to solve

$$\begin{cases} 2t - \pi \sin(t) = 0.....(1) \\ 2 - \pi \cos(t) = 2......(2) \end{cases}$$

From (2), we have $t = \frac{\pi}{2} + n\pi$, $n \in \mathbb{Z}$. And t also satisfies (1), we have $t = \frac{-\pi}{2}$ and $\frac{\pi}{2}$. Thus $\frac{dy}{dx}|_{t=\frac{-\pi}{2}} = \frac{-\pi}{2}$ and $\frac{dy}{dx}|_{t=\frac{\pi}{2}} = \frac{\pi}{2}$. Hence the tangent lines at the point $(0, 2)$ are $y - 2 = \frac{-\pi}{2}x$ and $y - 2 = \frac{\pi}{2}x$.

(Grading Rule:

1. Each correct t value earns 0.5 point.
2. Each slope of tangent line earns 1 point.
3. Each correct equation of tangent line earns 0.5 point.)

(c)

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \text{ (1 point)} \\ &= \frac{2\pi \cos t - \pi^2}{(2 - \pi \cos t)^3}. \text{ (2 points)} \\ \frac{d^2y}{dx^2}\Big|_{t=\pi/3} &= \frac{2\pi \cos(\frac{\pi}{3}) - \pi^2}{(2 - \pi \cos(\frac{\pi}{3}))^3} < 0 \text{ (1 point)} \\ &\Rightarrow \text{The curve near } t = \pi/3 \text{ is concave downward. (1 point)} \end{aligned}$$

(d)

$$
\begin{aligned}
A &= \int_0^{2\pi} y dx - 2\cdot 2\pi \\
&= \int_{\pi/2}^{\pi} (2-\pi\cos(t))(2-\pi\cos(t))dt - 4\pi \\
&= \int_{\pi/2}^{\pi} (4-4\pi\cos(t)+\pi^2\cos^2(t))dt - 4\pi \\
&= \int_{\pi/2}^{\pi} \pi^2\frac{1+\cos(2t)}{2} - 4\pi\cos(t)dt - 2\pi \\
&= \frac{\pi^2}{2}(t+\frac{1}{2}\sin(2t)) - 4\pi\sin(t)|_{t=\pi/2}^{t=\pi} - 2\pi \\
&= \frac{\pi^3}{4} + 2\pi
\end{aligned}
$$

(Grading Rules:

(1) 5 points: final answer is $\frac{\pi^3}{4} + 6\pi$.

(2) 3 points: $\int (2-\pi\cos t)^2 dt = 4t - 4\pi\sin t + \frac{\pi^2}{2}(t+\frac{1}{2}\sin(2t)) + C$.

(3) 2 points: if you remember subtract 4π for the area.

(4) 2 points for only the integral $A = \int_{\pi/2}^{\pi}(2-\pi\cos(t))(2-\pi\cos(t))dt$ is correct.

(5) 1 point for only the integral $A = \int_0^{2\pi} ydx$ is correct.

(6) 1 point for the integral $A = \int (2-\pi\cos(t))(2-\pi\cos(t))dt$, the interval is wrong.

2. (14 %) The graph $z = f(x, y)$ of the differentiable function f has $2x - 3y + z = 4$ as its tangent plane at the point $(0, 0, 4)$. The graph $z = g(x, y)$ of the differentiable function g has $x + 2y - z = 3$ as its tangent plane at the point $(0, 0, -3)$. Answer the following questions.

(a) (6 %) Determine the values: $f(0,0)$, $f_x(0,0)$, $f_y(0,0)$, $g(0,0)$, $g_x(0,0)$, $g_y(0,0)$.

(b) (2 %) Use the linearization of f at $(0,0)$ to estimate $f(0.1, -0.1)$.

(c) (6 %) Let $h(u, v) = ue^{-2v}$ and $u = f(x, y)$, $v = g(x, y)$. Use the Chain Rule to find the partial derivative

$$\frac{\partial}{\partial x} h\big(f(x,y), g(x,y)\big) \quad \text{at} \quad x = 0, y = 0.$$

Solution:

(a)

$$f(0,0) = 4,\ f_x(0,0) = -2,\ f_y(0,0) = 3,\ g(0,0) = -3,\ g_x(0,0) = 1,\ g_y(0,0) = 2$$

□

(b)

$$L(x,y) = -2x + 3y + 4$$
$$f(0.1, -0.1) \approx L(0.1, -0.1) = 3.5$$

□

(c)

$$\frac{\partial}{\partial x} h\left(f(x,y), g(x,y)\right) = \frac{\partial h}{\partial u}\frac{\partial f}{\partial x} + \frac{\partial h}{\partial v}\frac{\partial g}{\partial x}$$
$$\frac{\partial h}{\partial u} = e^{-2v}, \quad \frac{\partial h}{\partial v} = -2ue^{-2v}$$

At $(x, y) = (0, 0)$,

$$u = 4,\ v = -3,\ \frac{\partial h}{\partial u} = e^6,\ \frac{\partial h}{\partial v} = -8e^6$$
$$\frac{\partial}{\partial x} h\left(f(x,y), g(x,y)\right) = (e^6)\cdot(-2) + (-8e^6)\cdot(1) = -10e^6$$

□

Grading guideline:

(a) (1%) each. Note that they need to use their answer in (a) for (b) and (c).

(b) They can use the tangent plane or answer from (a).

(c) (3%) for Chain Rule, (2%) for finding the correct partials of h along with the values of u, v, (1%) for putting the numbers together.

3. (10 %) Let $f(x, y) = \ln(x - y + 1) - 2x + 2y$.

(a) (5 %) Find directions (unit vectors), $\mathbf{u}$, such that $D_{\mathbf{u}}f(1, 1) = -1$.

(b) (2 %) Is there any direction (a unit vector), $\mathbf{v}$, such that $D_{\mathbf{v}}f(1, 1) = 2$? Please explain your reason.

(c) (3 %) Find the tangent line of the level curve $f(x, y) = 0$ at $(1, 1)$.

Solution:

(a) Set $\mathbf{u} = \langle a, b\rangle$ where $a^2+b^2 = 1$ (1%). We compute that $f_x(x, y) = \dfrac{1}{x - y + 1} - 2$, $f_x(1, 1) = \dfrac{1}{1 - 1 + 1} - 2 = -1$ (1%), $f_y(x, y) = \dfrac{-1}{x - y + 1} + 2$ and $f_y(1, 1) = \dfrac{-1}{1 - 1 + 1} + 2 = 1$ (1%). Then we have

$$-1 = D_{\mathbf{u}}f(1, 1) = \langle f_x(1, 1), f_y(1, 1)\rangle \cdot \langle a, b\rangle = -a + b. \text{ (1\%)}$$

Substitute $b = a - 1$ into $a^2 + b^2 = 1$, we have $a = 1, 0$ and $b = 0, -1$. So $\mathbf{u} = \langle 1, 0\rangle$ or $\mathbf{u} = \langle 0, -1\rangle$ (1%).

(b) Since the maximum value of $D_{\mathbf{v}}f(1, 1)$ in any direction $\mathbf{v}$ is $|\nabla f(1, 1)| = \sqrt{2}$, there is no direction $\mathbf{v}$ such that $D_{\mathbf{v}}f(1, 1) = 2$ (2%).

(c) The equation of the tangent line is

$$f_x(1, 1)(x - 1) + f_y(1, 1)(y - 1) = 0 \;\Rightarrow\; -(x - 1) + (y - 1) = y - x = 0 \text{ (3\%)}.$$

4. (16 %) Let $f(x,y) = (2-y)(x^2+4y^2)$.

(a) (8 %) Find all critical points of $f(x,y)$ and determine which is a saddle point or gives a local maximum or local minimum.

(b) (8 %) Let $D = \{(x,y) \mid x^2+y^2 \le 4\}$. Find the absolute maximum and minimum values of $f(x,y)$ on D.

Solution:

(a) We compute that $f_x(x,y) = (2-y)2x$ (0.5%), $f_y(x,y) = -(x^2+4y^2)+(2-y)8y = -x^2+16y-12y^2$ (0.5%), $f_{xx}(x,y) = (2-y)2$ (0.5%),$f_{yy}(x,y) = 16-24y$ (0.5%) and $f_{xy}(x,y) = -2x$ (0.5%).
To solve $f_x(x,y) = 0$, we have $x = 0$ or $y = 2$ (0.5%).
For $x = 0$, to solve $f_y(0,y) = 16y - 12y^2 = 0$, we have $y = 0, 4/3$. For $y = 2$, we know $f_y(x,2) = -x^2 - 16 < 0$. So there are two critical points $(0,0)$ and $(0,4/3)$ (1%).
For $(0,0)$, we compute

$$D(0,0) = f_{xx}(0,0)f_{yy}(0,0) - [f_{xy}(0,0)]^2 = 4 \cdot 16 - 0^2 > 0,\ f_{xx}(0,0) = 4 > 0 \ (1\%)$$

For $(0,4/3)$, we compute

$$D(0,4/3) = f_{xx}(0,4/3)f_{yy}(0,4/3) - [f_{xy}(0,4/3)]^2 = \frac{4}{3}\cdot(-16) - 0^2 < 0,\ (1\%)$$

By second derivative test, $f(0,0) = 0$ is a local minimum and $(0.4/3)$ is a saddle point (2%).

(b) (M1) For $x^2+y^2 < 4$, f has two critical points $(0,0)$ and $(0,4/3)$ (1%). Then we evaluate $f(0,0) = 0$ and $f(0,4/3) = \dfrac{128}{27}$ (1%).
On $x^2+y^2 = 4$, we have $f(x,y) = (2-y)(4+3y^2)$ (1%). Then we consider $g(y) = (2-y)(4+3y^2)$, $-2 \le y \le 2$. (1%) To solve $g'(y) = -4+12y-9y^2 = 0$, we have $y = 2/3$ (1.5%). Then we evaluate $g(2) = 0$, $g(-2) = 64$ and $g(2/3) = 64/9$ (1.5%). So the absolute maximum of $f(x,y)$ on D is 64 and the absolute minimum of $f(x,y)$ on D is 0 (1%).

(M2) For $x^2+y^2 < 4$, f has two critical points $(0,0)$ and $(0,4/3)$ (1%). Then we evaluate $f(0,0) = 0$ and $f(0,4/3) = \dfrac{128}{27}$(1%).
On $x^2+y^2 = 4$, we have $f(x,y) = (2-y)(4+3y^2)$(1%). Set $x = 2\cos\theta$, $y = 2\sin\theta$, $0 \le \theta \le 2\pi$. Then we consider $g(\theta) = (2-2\sin\theta)(4+12\sin^2\theta)$.(1%) To solve $g'(\theta) = -8\cos\theta(1-6\sin\theta+9\sin^2\theta) = 0$, we have $\theta = \pi/2, 3\pi/2, \sin^{-1}(1/3)$(1.5%). Then we evaluate $g(\pi/2) = 0$, $g(3\pi/2) = 64$ and $g(\sin^{-1}(1/3)) = 64/9$(1.5%). So the absolute maximum of $f(x,y)$ on D is 64 and the absolute minimum of $f(x,y)$ on D is 0(1%).

(M3) For $x^2+y^2 < 4$, f has two critical points $(0,0)$ and $(0,4/3)$ (1%). Then we evaluate $f(0,0) = 0$ and $f(0,4/3) = \dfrac{128}{27}$ (1%).
On $x^2+y^2 = 4$, we set $g(x,y) = x^2+y^2$. By method of Lagrange multiplier, we consider

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 4 \end{cases} \Rightarrow \begin{cases} (2-y)2x = \lambda(2x) \\ -x^2+16y-12y^2 = \lambda(2y) \\ x^2+y^2 = 4 \end{cases} \ (1\%)$$

If $\lambda = 0$, from first two equations, we have $(x,y) = (0,0)$ or $(x,y) = (0,4/3)$ which contradict third equation. (1%)
For $\lambda \ne 0$, from first equation, we have $x = 0$ or $y = 2-\lambda$.(1%)
Substitute $x = 0$ into second and third equations, we obtain $y = \pm 2$.(0.5%)
Substitute $y = 2-\lambda$ into second equation and use third equation, we have

$$(2-\lambda)^2 - 4 + 16(2-\lambda) - 12(2-\lambda)^2 = 2\lambda(2-\lambda) \Rightarrow 9\lambda^2 - 24\lambda + 16 = 0 \Rightarrow \lambda = \frac{4}{3}$$

So $(x,y) = (\pm 4\sqrt{2}/3, 2/3)$.(0.5%) We evaluate $f(\pm 4\sqrt{2}/3, 2/3) = 64/9$, $f(0,-2) = 64$ and $f(0,2) = 0$.(1%) So the absolute maximum of $f(x,y)$ on D is 64 and the absolute minimum of $f(x,y)$ on D is 0.(1%)

5. (12 %) Compute the integrals.

(a) (6 %) $\int_0^2 \int_{\frac{y}{2}}^1 \frac{1}{1+x^4}\, dx\, dy$.

(b) (6 %) $\int_0^4 \int_0^{\sqrt{16-x^2}} \int_0^{\sqrt{16-x^2-y^2}} z^2\, dz\, dy\, dx$.

Solution:

(a)

$$\int_0^2 \int_{y/2}^1 \frac{1}{1+x^4}\ dx\ dy = \int_0^1 \int_0^{2x} \frac{1}{1+x^4}\ dy\ dx$$

$$= \int_0^1 \frac{2x}{1+x^4}\ dx = \int_0^1 \frac{du}{1+u^2}$$

(with $u = x^2$)

$$= \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4}$$

□

(b)

$$\int_0^4 \int_0^{\sqrt{16-x^2}} \int_0^{\sqrt{16-x^2-y^2}} z^2\ dz\ dy\ dx$$

$$= \frac{1}{3} \int_0^4 \int_0^{\sqrt{16-x^2}} (16 - x^2 - y^2)^{3/2}\ dy\ dx$$

$$= \frac{1}{3} \int_0^{\pi/2} \int_0^4 (16 - r^2)^{3/2} \cdot r\ dr\ d\theta$$

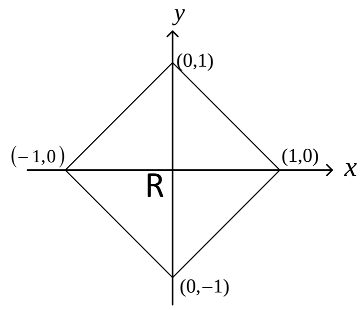
$$= \frac{\pi}{6} \cdot \frac{1}{5}(16)^{5/2} = \frac{\pi \cdot 2^9}{15} = \frac{512\pi}{15}$$

□

Grading guideline:

(-2%) for each minor mistake. (-4%) for each major mistake. Don't take points off for simplification errors.

6. (12 %) Evaluate $\displaystyle\iint_R \cos(x-y)e^{x+y}\,dA$, where $R = \{(x,y)\,\big|\,|x|+|y|\le 1\}$.



Solution:

Let $u = x+y$, $v = x-y$. Then $x = \dfrac{u+v}{2}$ and $y = \dfrac{u-v}{2}$. (+2)

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}. \qquad (+2)$$

(If students compute Jacobian as $\dfrac{\partial(u,v)}{\partial(x,y)}$, they lose 2 points.)

The corresponding region of R in the uv-plane is

$$D = \{(u,v)\mid -1\le u\le 1,\ -1\le v\le 1\} \qquad (+2)$$

Hence

$$\iint_R \cos(x-y)e^{x+y}\,dA = \iint_D \cos(v)\,e^u \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv \qquad (+2)$$

$$= \int_{-1}^{1}\int_{-1}^{1} \cos(v)\,e^u \frac{1}{2}\,du\,dv = \frac{e-e^{-1}}{2}\int_{-1}^{1}\cos v\,dv \quad (\text{+2 for integrating } e^u)$$

$$= \frac{e-e^{-1}}{2}(\sin 1 - \sin(-1)) = (e-e^{-1})\sin 1 \quad (\text{+2 for integrating } \cos v)$$

(If students forget to multiply the integrand with Jacobian when changing variables, they lose 2 points. However, they can still get credits for correctly integrating e^u, $\cos v$.)

7. (16 %) The continuous random variable X represents the amount of rainfall (in inches) in a garden in April and has probability density function $f_X(x) = \frac{3}{4}x(2-x)$, for $0 \le x \le 2$. The random variable Y represents the amount of manual watering (in inches) in the garden and has probability density function $f_Y(y) = \frac{1}{2}$, for $0 \le y \le 2$. Assume that X and Y are independent and have joint probability density function $f_{XY}(x,y) = \frac{3}{8}x(2-x)$ for $x, y \in [0,2]\times[0,2]$. Let $Z = X + Y$.

(a) (7 %) Compute the probability $P(Z \ge 3)$ and $P(Z \le 3)$.

(b) (7 %) Compute the probability $P(Z \le z)$ where z is a constant and $0 \le z \le 2$.

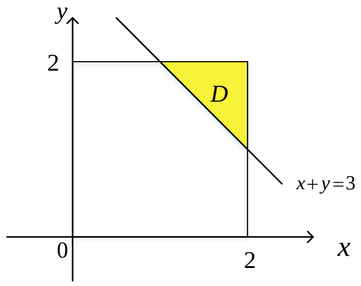
(c) (2 %) Compute the probability density function of Z which is $\frac{d}{dz}P(Z \le z)$, for $0 < z < 2$.

Solution:

(a)

$$P(Z \ge 3) = \iint_D f_{XY}(x,y)\,dA = \iint_D \frac{3}{8}x(2-x)\,dA$$

where $D = \{(x,y) \mid (x,y) \in [0,2]\times[0,2], \text{ and } x + y \ge 3\}$. (+2)



We can describe D as a type II region:

$$D = \{(x,y) \mid 1 \le y \le 2,\ 3 - y \le x \le 2\}.$$

Hence

$$P(Z \ge 3) = \int_1^2 \int_{3-y}^2 \frac{3}{8}x(2-x)\,dxdy \qquad (+2)$$

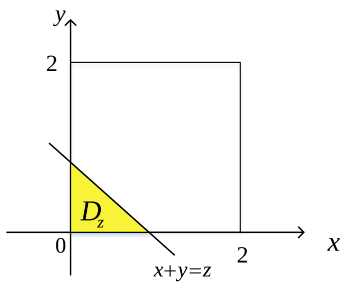
$$= \int_1^2 \frac{1}{2} - \frac{3}{8}(3-y)^2 + \frac{1}{8}(3-y)^3\,dy = \frac{3}{32}. \quad (+2)$$

Moreover, $P(Z \le 3) = 1 - P(Z \ge 3) = 1 - \frac{3}{32} = \frac{29}{32}$. (+1)

(Students may describe D as a type I region: $D = \{(x,y) \mid 1 \le x \le 2,\ 3 - x \le y \le 2\}$. Then $P(Z \ge 3) = \int_1^2 \int_{3-x}^2 \frac{3}{8}x(2-x)\,dydx$.)

(b) For $0 \le z \le 2$, $P(Z \le z) = \iint_{D_z} \frac{3}{8}x(2-x)\,dA$, where

$$D_z = \{(x,y) \mid (x,y) \in [0,2]\times[0,2], \text{ and } x + y \le z\}. \qquad (+2)$$



We can describe D_z as the a type II region:

$$D_z = \{(x,y) \mid 0 \le y \le z,\ 0 \le x \le z - y\}.$$

Hence

$$P(Z \le z) = \int_0^z \int_0^{z-y} \frac{3}{8}x(2-x)\,dxdy \qquad (+2)$$

$$= \int_0^z \frac{3}{8}(z-y)^2 - \frac{1}{8}(z-y)^3\,dy = \frac{z^3}{8} - \frac{z^4}{32}. \quad (+3)$$

(Students may describe D_z as a type I region: $D_z = \{(x,y) \mid 0 \le x \le z,\ 0 \le y \le z-x\}$. Then $P(Z \le z) = \int_0^z \int_0^{z-x} \frac{3}{8}x(2-x)\,dydx$.)

(c) For $0 < z < 2$, the probability density function of Z is $\frac{d}{dz}P(Z \le z) = \frac{3}{8}z^2 - \frac{1}{8}z^3$. (+2)